# ARITHMETIC MEANS OF FOURIER-STIELTJES-SINE-COEFFICIENTS

GÜNTHER GOES[1]

The theorem stated below contains improvements of statements of Hardy [4] and Kinukawa and Igari [6] and an interesting supplement to theorems of Fejér [1], [11, p. 107, 9.3] and Wiener [9], [11, p. 108, 9.6].

DEFINITIONS. Let $L$ be the space of Fourier coefficients of Lebesgue integrable functions and $dV$ the space of Fourier-Stieltjes-coefficients. Unambiguously let $L$ and $dV$ also denote the corresponding spaces of Fourier series and Fourier-Stieltjes-series respectively. Furthermore let

$$f \equiv \sum_{j=1}^{\infty} b_j \sin jt, \qquad \tilde{f} \equiv \sum_{j=1}^{\infty} b_j \cos jt.$$

In the following let $E$ and $E_1$ be $BK$-spaces [2, p. 350] contained in $dV$. Then $E_s$ and $E_c$ are the spaces in $E$ of sine- and cosine-coefficients respectively and $\tilde{E}=E_s \cap E_c$. If $b=\{b_j\}\in\tilde{E}$, $\|b\|_{E_s}=\|f\|_E$, $\|b\|_{E_c}=\|\tilde{f}\|_E$, then $\tilde{E}$ is a $BK$-space with the norm $\|b\|_{\tilde{E}}=\|b\|_{E_s}+\|b\|_{E_c}$ [10, p. 472]. Let

$$B_n = n^{-1}\sum_{j=1}^{n} b_j, \qquad B = \{B_n\},$$

and denote by $T_H$ the mapping $b\rightarrow B$. $T_H\in(E, E_1)$ means $b\in E$ implies $B\in E_1$.

STATEMENTS. Hardy [4] proved $T_H\in(L_c, L_c)$ and $T_H\in(L_s, L_s)$ is also true [7], [3, Theorem 27]. Even the following can be proved.

THEOREM. *If* $\sum_{j=1}^{\infty} b_j \sin jt$ *is a Fourier-Stieltjes-series, then* $\sum_{n=1}^{\infty} B_n \sin nt\in L$ *and* $\sum_{n=1}^{\infty} B_n \cos nt\in L$, *where* $B_n=n^{-1}\sum_{j=1}^{n} b_j$. *Or in symbols*: $T_H\in(dV_s, \tilde{L})$.

SKETCH OF PROOF. $T_H$ is a linear bounded transformation from $L_s$ into $L_s$ [10, p. 471] and $T_H\in(L_s, L_s)$ implies $\sup_n \|T_n\|<\infty$ [2, Theorem 4.4] where $T_n = \sup_{\|f\|_L\leq 1}\|T_nf\|_L$ and $T_nf = \sum_{j=1}^{n}(1-j/(n+1))B_j \sin jt$.

Since $dV$ is a norm determining manifold in $L$ and since $\|f\|_L =\|f\|_{dV}$ for $f\in L$ we have also $\|T_n\|=\sup_{\|f\|_{dV}\leq 1}\|T_nf\|_{dV}=O(1)$ $(n\rightarrow\infty)$

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and therefore $T_H \in (dV_s, dV_s)$ [2, Theorem 4.5]. (Correspondingly we get $T_H \in (dV_c, dV_c)$ but this is of no interest here.)

By Kinukawa and Igari [6, p. 274] we have $T_H \in (L_s, L_c)$ and since $T_H \in (L_s, L_s)$ we have $T_H \in (L_s, \tilde{L})$. The proof that $T_H \in (L_s, \tilde{L})$ implies $T_H \in (dV_s, d\tilde{V})$ is exactly the same as the proof that $T_H \in (L_s, L_s)$ implies $T_H \in (dV_s, dV_s)$. Since $d\tilde{V} = \tilde{L}$ [8; 11, p. 285] we have $T_H \in (dV_s, \tilde{L})$.

REMARKS. 1. Let $V$ be the space of Fourier coefficients of functions of bounded variation. From the fact that $b \in \tilde{V}$ implies $\sum_{j=1}^{\infty} |b_j| < \infty$ [5; 11, p. 286] it follows with our theorem that $b \in dV_s$ implies $\sum_{n=1}^{\infty} n^{-2} |\sum_{j=1}^{n} b_j| < \infty$.

2. As remarked by Loo [7, p. 270] we have $T_H \notin (L_c, L_s)$.

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