A NEW PROOF OF A COMPARISON THEOREM FOR ELLIPTIC EQUATIONS

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The classical Sturmian comparison theorem deals with solutions u, v of two ordinary differential equations.

$$\frac{d}{dx}\left(a\frac{du}{dx}\right) + fu = 0,$$

$$\frac{d}{dx}\left(b\frac{dv}{dx}\right) + gv = 0,$$

where a and b are positive and continuously differentiable on a closed interval I and f and g are continuous on I. The theorem states that if $a-b\geqq 0$ and $f\leqq g$ on I and x_1, x_2 are zeros of u in I, then v has a zero in $[x_1, x_2]$.

This theorem was generalized to self-adjoint second order elliptic partial differential equations by Hartman and Wintner [1]. The purpose of this note is to present a new and simple proof of this generalization.

Let u and v be solutions of the elliptic equations

$$\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + fu = 0, \qquad a_{ij} = \bar{a}_{ji}, \tag{1}$$

$$\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(b_{ij}\frac{\partial v}{\partial x_i}\right) + gv = 0, \qquad b_{ij} = \bar{b}_{ji} \tag{2}$$

in a bounded domain $R\subset E^n$. We assume that the a_{ij} and b_{ij} are of class C' and that f and g are real and continuous in $\bar{R}$. The ellipticity of (1) and (2) requires that the hermitean matrices $A=(a_{ij})$ and $B=(b_{ij})$ be positive definite in $\bar{R}$.

THEOREM. *If G is a bounded domain, $\bar{G}\subset R$, and*
(i) *$u=0$ on ∂G.*
(ii) *$A-B$ is non-negative definite in $\bar{G}$.*[1]
(iii) *$g\geqq f$ in $\bar{G}$.*
Then v must have a zero in $\bar{G}$.

PROOF. Since f is continuous in $\bar{R}$ we can choose a constant c so

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[1] This is equivalent to the condition "$B^{-1}-A^{-1}$ is non-negative definite" which is used in [1].

that $f(x)+c>0$ in $\bar{R}$. Let $F(x)=f(x)+c$. Define the operator

$$L=-\frac{1}{F}\sum\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_i}\right)+\frac{c}{F}\cdot$$

Equation (1) states that u is an eigenfunction of the boundary problem

$$\begin{aligned} Lu &= \lambda u, \\ u &= 0 \text{ on } \partial G \end{aligned} \tag{1'}$$

corresponding to the eigenvalue $\lambda=1$. Since the nodal lines of u divide G into a finite number of domains in which $u\neq 0$ it is sufficient to prove that the theorem holds for each such domain. In other words we may assume that $u\neq 0$ in G and that $\lambda_1=1$.

Suppose $v\neq 0$ in $\bar{G}$. Then we can choose a domain H satisfying $\bar{G}\subset H\subset R$ and $v\neq 0$ in H. Define

$$M=-\frac{1}{F}\sum\frac{\partial}{\partial x_j}\left(b_{ij}\frac{\partial}{\partial x_i}\right)+1-\frac{g}{F}.$$

Let n denote the exterior normal to ∂H. Then, by an appropriate choice of a function σ on ∂H, equation (2) states that v is an eigenfunction of the boundary value problem

$$\begin{aligned} Mv &= \nu v, \\ v+\sigma\frac{\partial v}{\partial n} &= 0 \text{ on } \partial H \end{aligned} \tag{2'}$$

corresponding to the eigenvalue $\nu=1$. Since $v\neq 0$ in H, $\nu_1=1$. Since $\bar{G}\subset H$, classical variational principles guarantee that the first eigenvalue of the boundary problem

$$\begin{aligned} Mw &= \mu w, \\ w &= 0 \text{ on } \partial G \end{aligned} \tag{2''}$$

will satisfy $\mu_1>\nu_1=1$. In particular, the assumption $v\neq 0$ in $\bar{G}$ has lead to the conclusion $\mu_1>\lambda_1$. We shall show that this conclusion is untenable.

Assuming

$$\int_G F\,|\,u\,|^2dx=\int_G F\,|\,w\,|^2dx=1$$

the minimal property of μ_1 yields

$$\mu_1 = \int_G F\bar{w}Mw\,dx \leqq \int_G F\bar{u}Mu\,dx.$$

Writing

$$M = -\frac{1}{F}\sum \frac{\partial}{\partial x_j}\left(b_{ij}\frac{\partial}{\partial x_i}\right) + \frac{c}{F} + \left(1 - \frac{g+c}{f+c}\right)$$

and noting that

$$1 - \frac{g+c}{f+c} \leqq 0 \text{ in } G,$$

$$\sum b_{ij}\xi_i\bar{\xi}_j \leqq \sum a_{ij}\xi_i\bar{\xi}_j \qquad \text{for all } (\xi_1, \cdots, \xi_n)$$

we have

$$\int_G F\bar{u}Mu\,dx = \int_G\left[\sum b_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \bar{u}}{\partial x_j} + c\,|\,u\,|^2 + F\left(1 - \frac{g+c}{f+c}\right)|\,u\,|^2\right]dx$$

$$\leqq \int_G\left[\sum a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \bar{u}}{\partial x_j} + c\,|\,u\,|^2\right]dx = \lambda_1.$$

Thus we conclude that $\mu_1 \leqq \lambda_1$ and that $v(x) = 0$ for some x in $\bar{G}$.

BIBLIOGRAPHY

1. P. Hartman and A. Wintner, *On a comparison theorem for self-adjoint partial differential equations of elliptic type*, Proc. Amer. Math. Soc. **6** (1955), 862.

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