A GENERALIZATION OF FEJÉR'S PRINCIPLE CONCERNING THE ZEROS OF EXTREMAL POLYNOMIALS[1]

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Dedicated to Professor Einar Hille.

In 1922 Fejér set forth [1] a principle which has shown itself highly useful, to the effect that a polynomial $p_n(z) \equiv (z-\alpha_1)(z-\alpha_2) \cdots (z-\alpha_n)$ which minimizes any classical norm in the complex plane such as

$$\max |p_n(z)|, \qquad z \text{ on } E, \tag{1}$$

$$\sum_{k=1}^{m} |p_n(z_k)|^p, \qquad E: \{z_1, z_2, \cdots, z_m\}, \qquad p > 0, \tag{2}$$

$$\int_E |p_n(z)|^p |dz|, \qquad E \text{ a Jordan arc or curve}, \qquad p > 0, \tag{3}$$

on a closed bounded point set E containing at least $n+1$ distinct points, must have all its zeros in the convex hull of E. More generally, the norms (1), (2), (3) may be replaced by any *monotonic* norm, namely any norm that decreases whenever the polynomial $p_n(z)$ is replaced by an *underpolynomial* $q_n(z) \equiv (z-\beta_1)(z-\beta_2) \cdots (z-\beta_n) \not\equiv p_n(z)$; the latter term requires

$$\begin{aligned} |q_n(z)| &< |p_n(z)| && \text{on } E \text{ where } p_n(z) \neq 0, \\ q_n(z) &= p_n(z) && \text{on } E \text{ where } p_n(z) = 0. \end{aligned} \tag{4}$$

Fejér's principle is readily proved; if the zero α_1 of $p_n(z)$ lies exterior to the convex hull of E, if α is the point of the convex hull nearest α_1, and if we set $\alpha_1' = (\alpha+\alpha_1)/2$, then the polynomial $q_n(z) \equiv (z-\alpha_1')p_n(z)/(z-\alpha_1)$ is an underpolynomial of $p_n(z)$ on E, so $p_n(z)$ cannot minimize any monotonic norm on E.

The object of the present note is to give what is essentially a generalization of Fejér's principle. It applies to the minimization of the difference or quotient of twc monotonic norms of a polynomial on two disjoint point sets:

It is especially appropriate that this paper should be dedicated to Professor Einar Hille, in view of his now classical work on the complex zeros of solutions of differential equations.

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THEOREM 1. *Let $F(\alpha)$ be a functional of the complex variable α and of the closed bounded sets E_1 and E_2, which decreases whenever simultaneously $|z_1-\alpha|$ decreases for all z_1 on E_1 and $|z_2-\alpha|$ increases for all z_2 on E_2; if $F(\alpha)$ is a minimum, then α cannot lie on a line separating E_1 and E_2.*

Suppose α lies on a line L separating E_1 and E_2; let α' lie on the perpendicular to L at α in the sense from α toward the side of L on which E_1 lies, so that the distance $\alpha\alpha'$ is less than the distance from L to E_1. By the Pythagorean theorem applied to suitable triangles whose legs are respectively parallel and perpendicular to L, it follows that $|z_1-\alpha'|<|z_1-\alpha|$ for all z_1 in E_1 and $|z_2-\alpha'|>|z_2-\alpha|$ for all z_2 in E_2, whence by the properties of $F(\alpha)$ we have $F(\alpha')<F(\alpha)$, so $F(\alpha)$ is not a minimum of the functional.

If there exists a line L separating E_1 and E_2, there exist an infinity of them, and each such line separates a largest set F_1 containing E_1 from a largest set F_2 containing E_2; Theorem 1 asserts that α lies in F_1 or F_2.

As an immediate illustration of Theorem 1, we formulate

THEOREM 2. *Let the point sets E_1 and E_2 be disjoint, let E_1 consist of more than n points, and let $\|P(z)\|_1$ and $\|P(z)\|_2$ be monotonic norms on E_1 and E_2 respectively of the polynomial $P(z)\equiv\prod_1^n(z-\alpha_j)$. Then no zero α_j of $P(z)$ can lie on a line separating E_1 and E_2 if $P(z)$ minimizes the functional*

$$F(\alpha_j) \equiv \|P(z)\|_1 - \|P(z)\|_2. \tag{5}$$

This functional clearly satisfies the conditions of Theorem 1. However, it may be pointed out that neither Theorem 1 nor Theorem 2 guarantees the existence or uniqueness of a minimum of the functional. For instance, let us choose E_1: $\{z=1, 2\}$, E_2: $\{z=-1, -2\}$, $n=1$, $P(z)\equiv z-\alpha$, $\|P(z)\|_1=[\max|P(z)|,\ z \text{ on } E_1]$, $\|P(z)\|_2=\exp[\max|P(z)|,\ z \text{ on } E_2]$; here the functional $F(\alpha)\equiv F(\alpha_j)$ defined by (5) has no minimum. With the same definitions of E_1, E_2, n, $P(z)$, and $\|P(z)\|_1$, let us set $\|P(z)\|_2=[\max|P(z)|,\ z \text{ on } E_2]$, $F(\alpha)\equiv F(\alpha_j)$ defined by (5); here min $F(\alpha)$ occurs for all $\alpha\geqq\frac{1}{2}$, $F(\alpha)=-3$.

Under the conditions of Theorem 1, it follows by Theorem 1 that if E_1 and E_2 lie on a line L_1, and if a point A of L_1 separates E_1 and E_2 on L_1, then $F(\alpha)$ can be a minimum only if α lies on L_1 but does not separate E_1 and E_2 on L_1.

Theorem 1 contains Fejér's principle, for we may choose $\|P(z)\|_1$ as any monotonic norm on E_1, $\|P(z)\|_2$ as zero for every $P(z)$ and E_2, and define $F(\alpha)\equiv F(\alpha_j)$ by (5).

A lemma is convenient in establishing another result.

LEMMA 1. *Let the point α lie on a circle or line that separates the closed bounded point sets E_1 and E_2; then there exists a point α' near α such that we have*

$$\left|\frac{z_1-\alpha'}{z_2-\alpha'}\right| < \epsilon\left|\frac{z_1-\alpha}{z_2-\alpha}\right|, \qquad 0<\epsilon<1, \tag{6}$$

uniformly for all z_1 on E_1 and all z_2 on E_2.

Inequality (6) states that a certain cross-ratio is in modulus less than some ϵ, $0<\epsilon<1$. If the plane is transformed by a linear transformation of the complex variable that carries α to infinity, the given circle (or line) on which α lies is transformed into a line L separating the images E_1' and E_2' of E_1 and E_2. There exists a circle (near the line L) containing all of E_1' but no point of E_2' in its interior, so if α_1' denotes the center of this circle we have

$$\left|\frac{z_1'-\alpha_1'}{z_2'-\alpha_1'}\right| < \epsilon < 1$$

uniformly for all z_1' in E_1' and all z_2' in E_2', where ϵ is suitably chosen. The inverse of the preceding linear transformation carries α_1' into a point α' satisfying (6). It may be noticed that $\alpha'(\neq\alpha)$ can be chosen as near α as desired, in such a way that α and α' are mutually inverse in a circle separating E_1 and E_2, where α' and E_1 are separated by that circle from α and E_2. Also, if α lies at one of the two distinct intersections of two circles or lines L_1 and L_2 each of which separates E_1 and E_2, and if E_1 and E_2 lie respectively in two of the four regions into which L_1+L_2 separates the plane having no arc of L_1 or L_2 as part of their common boundary, then α' may be chosen near α on the circle of the coaxal family determined by L_1 and L_2 bisecting the angle between L_1 and L_2, in direction from α toward E_1.

We are now in a position to apply Lemma 1; Theorem 3 follows at once:

THEOREM 3. *Let $F(\alpha)$ be a function of α, and of the closed bounded point sets E_1 and E_2, which decreases whenever*

$$\frac{|z_1-\alpha|}{|z_2-\alpha|}$$

decreases simultaneously for all z_1 in E_1 and for all z_2 in E_2; if $F(\alpha)$ is a minimum, then α cannot lie on a circle or line separating E_1 and E_2.

If there exists a circle or line separating E_1 and E_2, there exist an infinity of them, each of which separates a largest point set F_1 containing E_1 from a largest point set F_2 containing E_2; Theorem 3 shows that α lies in F_1 or in F_2.

We state explicitly an application of Theorem 3:

THEOREM 4. *Let us set*

$$P(z) \equiv \prod_1^n (z - \alpha_j), \qquad F(\alpha_j) \equiv \frac{\|P(z)\|_1}{\|P(z)\|_2},$$

where the norms are Tchebycheff norms (with positive weight functions $\mu_1(z)$ and $\mu_2(z)$) on the closed bounded disjoint point sets E_1 and E_2, where E_1 contains more than n points; if $F(\alpha_j)$ is a minimum, α_j cannot lie on a circle or line separating E_1 and E_2.

If a zero, say α_1, of $P(z)$ lies on a line or circle separating E_1 and E_2, and if $F(\alpha_1)$ is a minimum, the following algebraic inequalities result from Lemma 1:

$$F(\alpha_1') \equiv \frac{\max\left[\mu_1(z_1)\,|z_1 - \alpha_1'| \cdot \prod_2^n |z_1 - \alpha_k|,\ z_1 \text{ on } E_1\right]}{\max\left[\mu_2(z_2)\,|z_2 - \alpha_1'| \cdot \prod_2^n |z_2 - \alpha_k|,\ z_2 \text{ on } E_2\right]}$$

$$\leqq \frac{\max\left[\mu_1(z_1) \prod_1^n |z_1 - \alpha_k|\right] \cdot \max\left|\dfrac{z_1 - \alpha_1'}{z_1 - \alpha_1}\right|}{\max\left[\mu_2(z_2) \prod_1^n |z_2 - \alpha_k|\right] \cdot \min\left|\dfrac{z_2 - \alpha_1'}{z_2 - \alpha_1}\right|} \leqq \epsilon \cdot F(\alpha_1),$$

$0 < \epsilon < 1$, a contradiction that establishes Theorem 4.

It is not essential to suppose in Theorem 4 that Tchebycheff norms are used, provided the norms are homogeneous of the same degree, in the sense that for arbitrary positive continuous functions $\lambda_1(z)$ and $\lambda_2(z)$ on E_1 and E_2 respectively we have for some $\rho(>0)$

$$\|\lambda_1(z)P(z)\|_1 \leqq [\max \lambda_1]^\rho \|P(z)\|_1, \quad [\min \lambda_2]^\rho \|P(z)\|_2 \leqq \|\lambda_2(z)P(z)\|_2.$$

For instance suppose E_1 and E_2 are rectifiable Jordan arcs or curves, and the norms are (as below) weighted pth and qth power norms respectively, $p > 0$, $q > 0$; if a zero α_1 of $P(\alpha)$ lies on a line or circle separating E_1 and E_2, and if $F(\alpha_1)$ is a minimum, we have

$$F(\alpha_1') \equiv \frac{\left[\int_{E_1} \mu_1(z_1) \,|\, z_1 - \alpha_1' \,|^p \cdot \prod_2^n \,|\, z_1 - \alpha_k \,|^p \,|\, dz_1 \,|\right]^{1/p}}{\left[\int_{E_2} \mu_2(z_2) \,|\, z_2 - \alpha_1' \,|^q \cdot \prod_2^n \,|\, z_2 - \alpha_k \,|^q \,|\, dz_2 \,|\right]^{1/q}}$$

$$\leqq \frac{\left[\int_{E_1} \mu_1(z_1) \prod_1^n \,|\, z_1 - \alpha_k \,|^p \,|\, dz_1 \,|\right]^{1/p} \cdot \max \left|\frac{z_1 - \alpha_1'}{z_1 - \alpha_1}\right|}{\left[\int_{E_2} \mu_2(z_2) \prod_1^n \,|\, z_2 - \alpha_k \,|^q \,|\, dz_2 \,|\right]^{1/q} \cdot \min \left|\frac{z_2 - \alpha_1'}{z_2 - \alpha_1}\right|}$$

$$\leqq \epsilon F(\alpha_1), \qquad 0 < \epsilon < 1,$$

a contradiction as before.

It may be noticed that Theorems 2 and 4 overlap to a considerable extent; on the one hand, Theorem 2 refers to separation of E_1 and E_2 only by a line rather than a line or circle; on the other hand, the norms of Theorem 2 are arbitrary monotonic norms, and if the functional of Theorem 4 is replaced by $\log F(\alpha_j) \equiv \log \|P(z)\|_1 - \log \|P(z)\|_2$, we have essentially the difference of two particular monotonic norms.

Under the conditions of Theorem 4, if E_1 and E_2 lie on a circle or line L, and if there exist circles separating them, then for the minimum functional all points α_j also lie on L. Whenever there exist disjoint circular regions containing E_1 and E_2 respectively, these regions contain all such α_j. If E_2 consists of a single point z_2, all zeros of the polynomial $P(z)$ minimizing $F(\alpha_j)$ lie in the convex hull of E_2 with respect to z_2; here if m (>0) denotes min $F(\alpha_j)$ we have $\|P(z)\|_2 \leqq \|P(z)\|_1/m$, which determines max $|P(z_2)|$ over all $P(z)$ with prescribed $\|P(z)\|_1$; in this case the remark concerning the location of the α_j is due to Szegö [2, §5] and Fekete [3, p. 344].

The preceding results have all been established by considering a local variation of α (or α_j); we proceed to consider the general question of a global variation of α, and related results concerning maxima and minima. We use the same notation for a circular region (closed interior or exterior of a circle, or half-plane) as for its boundary, and shall prove

LEMMA 2. *In the extended plane, let the circular regions C_1 and C_2 be disjoint. Let α' denote the null circle in C_1 belonging to the coaxal family determined by the circles C_1 and C_2, and let λ denote the ratio of the radius of the image of C_1 to the radius of the image of C_2 when α' is transformed to infinity.* (i) *If $\lambda > 3$, for all z_1 in C_1 and for all z_2 and α in C_2 inequality* (6) *holds uniformly with suitable ϵ (<1).* (ii) *If $\lambda \geqq 3$, for all z_1 in C_1 and for all z_2 and α in C_2 we have*

$$\left|\frac{z_1 - \alpha'}{z_2 - \alpha'}\right| \leqq \left|\frac{z_1 - \alpha}{z_2 - \alpha}\right|. \tag{7}$$

(iii) *If* $\lambda < 3$, *no point* α' *exterior to* C_2 *exists for which* (7) *is valid for all* z_1 *in* C_1 *and for all* z_2 *and* α *in* C_2, *but* (6) *with* α' *as previously defined is valid for all* z_1 *in* C_1, *for all* z_2 *in* C_2, *and for each fixed* α *in a suitable subregion* C_3 *of* C_2.

Since both (6) and (7) refer to the magnitude of a certain cross-ratio, it is no loss of generality to choose α' at infinity, and the circular regions C_1 and C_2 as $|z_1| \geqq R_1$ and $|z_2| \leqq R_2$ $(<R_1)$; we suppose too $|\alpha| \leqq R_2$. The restrictions already made imply $|z_2-\alpha| \leqq 2R_2$, $|z_1-\alpha| \geqq R_1-R_2$, so we have

$$\left|\frac{z_2 - \alpha}{z_1 - \alpha}\right| \leqq \frac{2R_2}{R_1 - R_2} = \epsilon \tag{8}$$

with $\epsilon<1$, thanks to our assumption $\lambda=R_1/R_2>3$. Inequality (8) is equivalent to (6) with $\alpha'=\infty$, which establishes (i). Part (ii) follows from (8) with $\epsilon=1$. The first part of (iii) is a consequence of the fact that with $1<\lambda<3$ and with the choices $z_1=R_1$, $\alpha=R_2$, $z_2=-R_2$ we have

$$\left|\frac{z_2 - \alpha}{z_1 - \alpha}\right| = \frac{2R_2}{R_1 - R_2} = \frac{2}{\lambda - 1} > 1;$$

for no choice of α' exterior to C_2 can we have

$$\left|\frac{z_1 - \alpha'}{z_2 - \alpha'}\right| \leqq 1$$

for all z_1 in C_1 and all z_2 in C_2, so (7) is impossible. For the second part of (iii), we notice that without the requirement $|\alpha| \leqq R_2$ we have $|z_2-\alpha| \leqq R_2+|\alpha|$, $|z_1-\alpha| \geqq R_1-|\alpha|$, and (6) is valid with $\epsilon<1$ provided we have $|\alpha| < R_1$ with

$$\frac{R_2 + |\alpha|}{R_1 - |\alpha|} \leqq \epsilon, \qquad |\alpha| \leqq \frac{\epsilon R_1 - R_2}{1 + \epsilon},$$

which is true for suitably chosen ϵ whenever $|\alpha| < (R_1-R_2)/2$, an inequality which obviously implies $|\alpha| < R_1$. It will be noted that the inequality $|\alpha| < (R_1-R_2)/2$ restricts α to the interior of a certain circular region C_3 interior to C_2; such a fixed α may be replaced by $\alpha'=\infty$ with (6) valid.

We remark incidentally that (6) and (7) are considered valid even with $\alpha' = z_1 = \infty$, as is reasonable in view of the invariance of cross-ratio under linear transformation.

The significance of Lemma 2 with the hypothesis of Theorem 4 is as follows. Let E_1 and E_2 lie respectively in the (disjoint) circular regions C_1 and C_2 of Lemma 2. If $F(\alpha_j)$ is a minimum, it follows by Theorem 4 that α_j cannot lie exterior to both C_1 and C_2. Lemma 2 implies that in case (i) the point α_j cannot lie in C_2, by precisely the application of (6) made in the proof of Theorem 4; in case (ii) with $\lambda = 3$, we may replace α_j in C_2 by α_j' in C_1 without increasing $F(\alpha_j)$; in case (iii) the point α_j for minimum $F(\alpha_j)$ cannot lie interior to a specified circular region C_3 which is a subregion of C_2 bounded by a circle of the coaxal family determined by the circles C_1 and C_2.

Both Theorem 2 and Theorem 4 refer to the separation of E_1 and E_2' by lines and circles, and are therefore strongly reminiscent of Bôcher's theorem [4, §4.2] to the effect that a finite point which lies on a line or circle separating the zeros and poles of a rational function $R(z)$ cannot be a zero of the derivative $R'(z)$. However, Bôcher's theorem is not related to any analogue of Lemma 2(i) and its application to Theorem 4. Indeed, Bôcher's theorem asserts that if two finite disjoint circular regions C_1 and C_2 contain respectively all zeros and all poles of the rational function $R(z)$ of degree n, and if the poles are all simple, then C_1 and C_2 contain each $n-1$ zeros of $R'(z)$; no zero of $R'(z)$ can be displaced from C_2.

Theorems 2 and 4 consider the difference and the quotient of the norms of a polynomial on E_1 and E_2; likewise the sum and the product of two monotonic norms on E_1 and E_2 may be considered, but the sum and product are themselves monotonic norms on $E_1 + E_2$, and it follows by Fejér's principle that all zeros of a minimizing polynomial lie in the convex hull of $E_1 + E_2$.

It is not to be supposed that the present note exhausts the significance of the methods used. The reader may consider for instance the following:

I. Suppose E_1 and E_2 are closed bounded disjoint point sets, and that E_1 contains at least $n+1$ points. If $p_n(z) \equiv (z-\alpha_1) \cdots (z-\alpha_n)$ and the functional

$$\frac{\max[\,|\,p_n(z)\,|,\ z \text{ on } E_1]}{\min[\,|\,p_n(z)\,|,\ z \text{ on } E_2]} \tag{9}$$

is least, the conclusion of Theorem 4 and possible application of Lemma 2 remain valid.

II. Under the same conditions on E_1 and E_2, and if E_2 also contains at least $n+1$ points, and if the functional (9) with $p_n(z)$ replaced by

$R(z) \equiv \prod_1^n (z-\alpha_k)/(z-\beta_k)$ is least, no α_k or β_k can lie on a circle or line separating E_1 and E_2; Lemma 2 also applies under suitable conditions.

III. Theorem 2 extends likewise to rational functions.

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