IDEMPOTENT MEASURES ON A COMPACT TOPOLOGICAL SEMIGROUP

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1. Introduction. Let S be a compact topological (Hausdorff) semigroup. Consider any probability measure ν regular on S. Some of the limit properties of the average of the convolution sequence

$$(1) \qquad \frac{1}{n}\sum_{j=1}^{n} \nu^{(j)} = \nu_n$$

were discussed in [5]. Let $\Sigma(\nu)$ be the support (or spectrum) of ν. One can just as well take S as the closure of $\bigcup_n (\Sigma(\nu))^n = S(\nu)$ since all the convolutions $\nu^{(j)}$ are concentrated on $S(\nu)$. It was shown that $\lim_{n\to\infty} \nu_n = \mu$ *exists in the sense that* $\lim_{n\to\infty} \int f d\nu_n = \int f d\mu$ *for every continuous* f on S. Further, μ is regular with support the kernel K (minimal two-sided ideal) of $S(\nu)$ and is idempotent, that is,

$$(2) \qquad \mu^{(2)} = \mu.$$

In [5] the definition of the convolution of two regular measures ν, μ on S was introduced as follows. Let $\mathcal{B} = \mathcal{B}(S)$ be the Borel field generated by the open sets of S. If $B \in \mathcal{B}(S)$, $\nu * \mu(B)$ was given by

$$\nu * \mu(B) = \int c_B(vu)(\nu \times \mu)(d(v, u))$$

where c_B is the characteristic function of the set B and $\nu \times \mu$ is the product measure generated by ν, μ on $S \times S$. This is not valid for all compact Hausdorff semigroups since $A_B = \{(v, u) \mid vu \in B\}$ may not be in the product Borel field $\mathcal{B}(S) \times \mathcal{B}(S)$ even though $B \in \mathcal{B}(S)$. It is valid for separable Hausdorff semigroups. However, one can generally introduce the convolution of two regular measures ν, μ on a compact Hausdorff semigroup S as follows. Given any continuous f on S, let

$$\int \left\{ \int f(vu)\nu(dv) \right\} \mu(du) = L(f).$$

This defines a continuous functional of the continuous functions on S. By the Riesz theorem [1] on such functionals, there is a regular

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measure on $\mathcal{B}(S)$ determined by $L(f)$

$$L(f) = \int f(u)(\nu * \mu)(du)$$

which we shall call the convolution $\nu * \mu$ of ν, μ. With this definition of the convolution $\nu * \mu$, the proofs of the results cited in [5] can be simply modified so as to hold for the case of a general compact Hausdorff semigroup S. Kernel semigroups K are rather special semigroups and are often referred to as completely simple semigroups [6]. Every compact completely simple semigroup can be represented as the product space $T\times X\times Y$ of a compact topological group T and compact Hausdorff spaces X, Y where the multiplication of two elements $s=(t, x, y)$, $s'=(t', x', y')$ is given by

$$ss' = (t, x, y)(t', x', y') = (t\phi(x, y')t', x', y) \tag{3}$$

with ϕ a continuous function on the product space $X\times Y$ into T [6]. We can therefore identify K with such a space $T\times X\times Y$ and the corresponding ϕ function.

In [5] it was shown that every idempotent measure μ on a finite completely simple semigroup K with support the whole semigroup is a $\bar{\mu}$ measure, that is, μ is a product measure

$$\bar{\mu} = \chi \times \alpha \times \beta$$

where χ is the normed Haar measure ($\chi(T)=1$) of the finite group T and α and β are probability measures on X and Y respectively. This note extends the above result to any compact Hausdorff semigroup.

THEOREM. *Let μ be a regular idempotent probability measure on a compact Hausdorff semigroup. Then μ has a completely simple subsemigroup K as its support. Further μ is a $\bar{\mu}$ measure so that if K has the representation $T\times X\times Y$ then on $\mathcal{B}(T)\times\mathcal{B}(X)\times\mathcal{B}(Y)$*

$$\mu = \chi \times \alpha \times \beta$$

where χ is the normed Haar measure of the group T and α, β are regular probability measures on X, Y respectively.

COROLLARY. *Given a regular measure ν on the compact Hausdorff semigroup S, the sequence of averaged convolution measures*

$$\nu_n = \frac{1}{n}\sum_{j=1}^{n} \nu^{(j)}$$

converges to a $\bar{\mu}$ measure with support the kernel of $S(\nu)$.

2. **Proof of the theorem.** We shall make use of a number of lemmas to prove the theorem. For convenience let

$$As^{-1} = \{s' \mid s's \in A\}. \tag{4}$$

LEMMA 1. *Let μ be a regular probability measure on the compact Hausdorff semigroup S. Then for each $s \in S$, the measure $\mu(As^{-1})$, $A \in \mathcal{B}(S)$, is regular on S.*

Here $\mathcal{B}(S)$ is the Borel field generated by the open sets on S. By the regularity of μ on S, given any $\epsilon > 0$, there is a closed set $C \subset As^{-1}$ such that $\mu(As^{-1} - C) < \epsilon$. Let $B = Cs$. By the continuity of the multiplicative operation, we see that B is closed. Further $B \subset A$. Thus $\mu((A - B)s^{-1}) < \epsilon$ since $C \subset Bs^{-1} \subset As^{-1}$.

LEMMA 2. *If $A \in \mathcal{B}(S)$ and μ is a regular measure on the compact Hausdorff semigroup S, $\mu(As^{-1})$ is a Borel measurable function of s.*

Let η denote the class of continuous functions f on S with $0 \leq f \leq 1$. Further, given any set A, let $\bar{A}$ denote the complement of A. First let O be an open set. Set

$$A_\alpha = \{s \mid \mu(Os^{-1}) > \alpha\}.$$

Now by the regularity of $\mu(\cdot s^{-1})$ (see [**1**; **2**])

$$\mu(Os^{-1}) = \sup_{f \in \eta; f = 0 \text{ on } \bar{O}} \int f(u)\mu(dus^{-1}) = \sup_{f \in \eta; f = 0 \text{ on } \bar{O}} \int f(us)\mu(du).$$

Given any $s \in A_\alpha$ there is an $\epsilon > 0$ such that $\mu(Os^{-1}) > \alpha + \epsilon$. But then there is a function $f_s \in \eta$ with $f_s = 0$ on $\bar{O}$ such that

$$\int f_s(us)\mu(du) > \alpha + \epsilon/2.$$

The set of points $\{z \mid \int f_s(uz)\mu(du) > \alpha + \epsilon/2\}$ is an open set containing s and is a subset of A_α. Hence A_α is open. This implies that $\mu(Os^{-1})$ is Borel measurable in s.

The open sets are a field. Further the class of sets $A \in \mathcal{B}(S)$ for which $\mu(As^{-1})$ is Borel measurable is a monotone class. Hence this class is a Borel field and must coincide with $\mathcal{B}(S)$ (see [**4**]).

LEMMA 3. *Given any set $A \in \mathcal{B}(S)$ and ν, μ regular probability measures on the compact Hausdorff semigroup S,*

$$\nu * \mu(A) = \int \nu(As^{-1})\mu(ds). \tag{5}$$

It is enough to show this for an open set O since it will then follow for general $A\in\mathcal{B}(S)$ by the regularity of $\nu*\mu$. Now

$$\nu*\mu(O)=\sup_{f\in\eta;f=0\text{ on }\bar{O}}\int\left\{\int f(vu)\nu(dv)\right\}\mu(du)$$
$$\leqq\int\left\{\sup_{f\in\eta;f=0\text{ on }\bar{O}}\int f(vu)\nu(dv)\right\}\mu(du)=\int\nu(Os^{-1})\mu(ds).$$

Consider any fixed $\epsilon>0$. Let $A_{k,n}=\{s\mid k/2^n\leqq\nu(Os^{-1})<(k+1)/2^n\}$, $k=0,1,\cdots,2^n$, with $2^{-n+2}<\epsilon$. Then

$$\left|\sum_{k=0}^{2^n}\frac{k}{2^n}\mu(A_{k,n})-\int\nu(Os^{-1})\mu(ds)\right|<\frac{1}{2^n}\cdot$$

There is a closed set $C_{k,n}\subset A_{k,n}$ with

$$\mu(A_{k,n}-C_{k,n})<\epsilon/2^{n+2},\qquad k=0,1,\cdots,2^n.$$

Given any $s\in C_{k,n}$, there is a continuous function $f_s\in\eta$, $f_s=0$ on $\bar{O}$, such that

$$\int f_s(vs)\nu(dv)>\nu(Os^{-1})-\frac{1}{2^n}\cdot$$

The set $B_s=\{z\mid\int f_s(vz)\nu(dv)>k/2^n-1/2^n\}$ is an open set containing s. Hence the sets B_s, $s\in C_{k,n}$, are an open covering of $C_{k,n}$. There is a finite subcovering $B_{s_1},\cdots,B_{s_j}$ of $C_{k,n}$. Let $f_{k,n}(s)=\max_{i=1,\cdots,j}f_{s_i}(s)$. Clearly

$$\int f_{k,n}(vz)\nu(dv)>\frac{k-1}{2^n}$$

for all $z\in C_{k,n}$. Further $f_{k,n}\in\eta$, $f_{k,n}=0$ on $\bar{O}$. In this way we obtain such a function $f_{k,n}$ for $C_{k,n}$, $k=0,1,\cdots,2^n$. Let $f(s)=\max_{k=0,1,\cdots,2^n}f_{k,n}(s)$. Then

$$\int\left\{\int f(vz)\nu(dv)\right\}\mu(dz)>\sum_{k=0}^{2^n}\frac{k-1}{2^n}\mu(C_{k,n})$$
$$>\int\nu(Os^{-1})\mu(ds)-\frac{2}{2^n}-\frac{\epsilon}{2}>\int\nu(Os^{-1})\mu(ds)-\epsilon$$

where $f\in\eta$, $f=0$ on $\bar{O}$. Since this holds for any $\epsilon>0$, we have the desired conclusion for open sets.

Let $F_n(F_n\subset A)$, $O_n(A\subset O_n)$ be nondecreasing and nonincreasing sequences of closed and open sets such that $\nu*\mu(F_n)$, $\nu*\mu(O_n)$

$\to \nu * \mu(A)$ as $n \to \infty$ for a fixed $A \in \mathcal{B}(S)$. The existence of such sequences follows from the regularity of $\nu * \mu$. But then

$$\begin{aligned} \nu * \mu(A) &= \lim_n \nu * \mu(O_n) = \lim_n \int \nu(O_n s^{-1})\mu(ds) \\ &\geqq \int \nu(As^{-1})\mu(ds) \geqq \lim_n \int \nu(F_n s^{-1})\mu(ds) \\ &= \lim_n \nu * \mu(F_n) = \nu * \mu(A). \end{aligned}$$

Thus Lemma 3 holds for general $A \in \mathcal{B}(S)$.

Suppose μ is an idempotent measure on S. Then

$$\begin{aligned} \mu(As^{-1}) &= \int \mu(As^{-1}s'^{-1})\mu(ds') = \int \mu(A(s's)^{-1})\mu(ds') \\ &= \int \mu(As'^{-1})\mu(ds's^{-1}). \end{aligned}$$

By Theorem 14 of [3] we already know that an idempotent probability measure must have a completely simple semigroup as its support. From this point on let us take S a compact completely simple semigroup with representation $T \times X \times Y$ and corresponding function ϕ.

Suppose μ is an idempotent measure on S with support S. Let

$$S_x = \{s \mid x(s) = x\}, \tag{6}$$

that is, S_x is the subset of points in S whose x coordinate $x(s)$ in representation ( ) of the semigroup is the fixed point x in X. Then $P(s, A) = \mu(As^{-1})$ is an idempotent Markov transition measure (see [4]) for $A \in \mathcal{B}(S_x)$, $s \in S_x$, that is

$$P(s, A) = \int_{S_x} P(s, ds')P(s', A). \tag{7}$$

$\mathcal{B}(S_x)$ is the Borel field on S_x induced by $\mathcal{B}(S)$. We shall call $B \in \mathcal{B}(S_x)$ an *invariant set* if

$$P(s, B) = 1 \tag{8}$$

for all $s \in B$. We say that S_x is *irreducible* if one cannot find two disjoint nonvacuous sets $A, B \in \mathcal{B}(S_x)$ such that

$$\begin{aligned} P(s, A) &\equiv 1 \quad \text{for all } s \in A, \\ P(s, B) &\equiv 1 \quad \text{for all } s \in B. \end{aligned} \tag{9}$$

LEMMA 4. *Let μ be a regular idempotent probability measure with support the compact completely simple semigroup S. Then S_x (for every $x \in X$) is irreducible with respect to $P(s, A) = \mu(As^{-1})$, $A \in \mathcal{B}(S_x)$, $s \in S_x$.*

Suppose S_x is not irreducible. Then there are two disjoint nonvacuous invariant sets A, $B \in \mathcal{B}(S_x)$. Both these invariant sets must be dense in S_x. For consider any invariant set A. Let s be any point of S_x. Consider any open neighborhood N_s of s and take a any point of A. $N_s a^{-1}$ is open by the continuity of the multiplicative operation. But then $P(a, N_s) = \mu(N_s a^{-1}) > 0$ and hence N_s contains an element of A. Thus A is dense in S_x.

Let a be an element of A. By the regularity of $\mu(\cdot a^{-1})$ on S_x there is a closed set $C \subset A$ such that $\mu(Ca^{-1}) > 1 - \epsilon > 0$. But $S_x - C$ is open in S_x and contains B. B is dense in S_x so that $S_x - C$ is all of S_x. However, this contradicts $\mu(Ca^{-1}) > 0$.

LEMMA 5. *Let μ be an idempotent probability measure with support the compact completely simple semigroup S. Then $P(s, A) = \mu(As^{-1})$ with $s \in S_x$, $A \in \mathcal{B}(S_x)$ is independent of s.*

Consider $P(s, A) = \mu(As^{-1})$ with $s \in S_x$, $A \in \mathcal{B}(S_x)$. Let

$$\bar{f}(s) = \int_{S_x} P(s, ds') f(s') \tag{10}$$

for f a bounded function on S_x measurable with respect to $P(s, \cdot)$ for every s. Since $P(s, \cdot)$ is an idempotent transition probability function (see (7))

$$\int_{S_x} P(s, ds') [\bar{f}(s') - f(s')] \equiv 0. \tag{11}$$

We shall call $f(s)$ *an almost invariant function* if the set $E_f = \{ s' \mid f(s') \neq \bar{f}(s') \}$ is of zero $P(s, \cdot)$ measure for every s. Consider a bounded function f such that

$$\bar{f}(s) \geqq f(s) \tag{12}$$

except possibly for a set G_f of zero $P(s, \cdot)$ measure for every s. Such a function is an almost invariant function since by (11) the set on which $f(s) \neq \bar{f}(s)$ is of zero $P(s, \cdot)$ measure for all s.

If f, g are almost invariant, then $\max(f, g)$ is almost invariant. The set of almost invariant functions is a linear space and is closed under bounded pointwise convergence. This implies that if f is an almost invariant function then for any fixed α the characteristic function $c_{A_\alpha}(s)$ of the set

$$A_\alpha = \{s \mid f(s) > \alpha\} \tag{13}$$

is almost invariant. For consider $g(s) = \max(f(s) - \alpha, 0)$ which is almost invariant. Let $h_n(s) = \min(ng(s) - 1, 0) + 1$. But $c_{A_\alpha}(s) = \lim_n h_n(s)$, that is, it is the limit of almost invariant functions and hence almost invariant. Thus, except for a set E which is of $P(s, \cdot)$ measure zero for every s, $c_{A_\alpha}(s) = \bar{c}_{A_\alpha}(s)$. If $s \in A_\alpha - E$ then

$$P(s, A_\alpha - E) = P(s, A_\alpha) = 1. \tag{14}$$

Similarly if $s \in \overline{A}_\alpha - E$ ($\overline{A}$ is the complement of A) then

$$P(s, \overline{A}_\alpha - E) = P(s, \overline{A}_\alpha) = 1. \tag{15}$$

Thus, if they are nonvacuous, $A_\alpha - E$ and $\overline{A}_\alpha - E$ are invariant sets.

Now consider setting $f(s) = P(s, A)$ for any fixed $A \in \mathcal{B}(S_x)$. Clearly $P(s, A)$ is almost invariant. The argument given above implies that there is an α such that

$$P(s, A) \equiv \alpha \tag{16}$$

for all s except those in a set E_A of $P(s, \cdot)$ measure zero for every s. But then

$$P(s, A) = \int_{S_x} P(s, ds')P(s', A) = \int_{\bar{E}_A} P(s, ds')P(s', A) \equiv \alpha \tag{17}$$

for all s, whether outside E_A or in E_A. The proof of Lemma 5 is complete.

We are now ready to finish the proof of the theorem. Let us look at

$$\mu((U \times V \times W)s^{-1}) \tag{18}$$

where $U \times V \times W$ is a product set with $U \in \mathcal{B}(T)$, $V \in \mathcal{B}(X)$, $W \in \mathcal{B}(Y)$. Notice that if (18) is positive we must have $x(s) \in V$. Now

$$(U \times V \times W)s^{-1} = \{s' \mid t(s') \in Ut(s)^{-1}\phi(x(s'), y(s))^{-1}, y(s') \in W\}. \tag{19}$$

Lemma 5 implies that (18) is independent of s for which $x(s) \in V$ and therefore by (19)

$$\mu((U \times V \times W)s^{-1}) = \mu((Ut \times V \times W)s^{-1}) \tag{20}$$

for all $t \in T$ if $x(s) \in V$. Expression (18) is zero if $x(s) \notin V$. But this implies that

$$\mu((U \times V \times W)s^{-1}) = \chi(U)\mu((T \times V \times W)s^{-1}) \tag{21}$$

where χ is the normed Haar measure of T (the support of (18) with V, W fixed and $\mu(T \times V \times W) > 0$ is all of T). However

$$\mu((T \times V \times W)s^{-1}) = \beta(W) \tag{22}$$

if $x(s) \in V$ and zero otherwise. Now

$$\mu((U \times V \times W)) = \int \mu((U \times V \times W)s^{-1})\mu(ds)$$

$$= \int_{x(s)\in V} \chi(U)\beta(W)\mu(ds) = \chi(U)\alpha(V)\beta(W).$$

The proof of the theorem is complete.

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