ON SOME OPEN QUESTIONS CONCERNING STRICTLY SINGULAR OPERATORS[1,2]

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T. Kato introduces the following concept of a strictly singular operator in [3].

DEFINITION. Let X and Y be Banach spaces and let T be a bounded linear operator mapping X into Y. T is said to be strictly singular if given any infinite dimensional subspace M of X, T restricted to M is not an isomorphism (i.e., linear homeomorphism).

In particular, every compact operator is strictly singular. Kato proceeds to show that the space $\mathcal{S}$ of strictly singular operators possesses some of the important features of the space of compact operators, e.g., $\mathcal{S}$ is a closed subspace of the space of bounded linear operators from X to Y. If $X=Y$, then $\mathcal{S}$ is a closed ideal. If X and Y are Hilbert spaces, then every strictly singular operator from X to Y is compact.

The following two questions, posed by Kato, are answered below.

(1) Is every strictly singular operator compact?

(2) Is the conjugate of a strictly singular operator strictly singular?

THEOREM.[3] (a) *Every bounded linear operator from l_2 to l_p or from l_p to l_2, $1 \leqq p \neq 2$, ∞ is strictly singular.*

(b) *Let X be a Banach space which does not contain an infinite dimensional reflexive subspace. Then every bounded linear operator mapping X into a reflexive space is strictly singular. Also any bounded linear operator mapping a reflexive space into X is strictly singular.*

REMARKS. (i) R. S. Phillips exhibited (unpublished) a strictly singular operator which is not compact. Theorem 1 shows that $\mathcal{S}$ can indeed be a "much larger" space than the space of compact linear operators.

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[2] Presented at the International Congress of Mathematicians in Stockholm on August 25, 1962.

[3] It has been pointed out to the authors by R. J. Whitley that essentially the same proof used in (a) can be applied to prove that every bounded linear operator from l_p to l_q, $1<p, q<\infty$, $p \neq q$, is strictly singular. As above, $\overline{M}$ is isomorphic to a subspace of l_q, hence by [1, Theorem 1, p. 194], l_q is isomorphic to a subspace of $\overline{M}$ and hence to a subspace of l_p. This, however, contradicts [1, Theorem 7, p. 205].

(ii) l_1 and c_0 are spaces which do not contain any infinite dimensional reflexive subspaces. To see this, suppose M were such a subspace of l_1. Since l_1 is not reflexive, it is not isomorphic to a subspace of M. But by [1, Theorem 1, p. 194], this can only be if M is finite dimensional, which is a contradiction. The argument for c_0 is the same.

Proof of part (a). Suppose $1<p\neq 2, \infty$. Let $T: l_2 \rightarrow l_p$ be bounded and linear. Suppose there exists an infinite dimensional subspace of M of l_2 such that T restricted to M has a bounded inverse. Then T restricted to $\overline{M}$, the closure of M in l_2, has a bounded inverse so that $\overline{M}$ is isomorphic to a subspace of l_p. Since $\overline{M}$ is a separable Hilbert space, it is equivalent to l_2. Thus l_2 is isomorphic to a subspace of l_p. This, however, contradicts a theorem due to Banach [1, Theorem 7, p. 205]. The proof that any bounded linear map from l_p to l_2, $1<p\neq 2$, ∞ is strictly singular is similar to the one just given. Part (b) includes the case for $p=1$.

Proof of part (b). Let X be as in (b). Suppose Y is reflexive. If $T: X \rightarrow Y$ is bounded and linear but not strictly singular, then X would contain a closed infinite dimensional subspace N which is isomorphic to a closed subspace of Y. Thus N is reflexive which cannot be. The same argument can be used to show that T: $Y \rightarrow X$ is strictly singular.

For question (2) we consider the

Example. Let T be a continuous linear operator mapping l_1 onto l_2. That such a map exists is a result of a theorem of Banach and Mazur [2, p. 111], which states that given any separable Banach space X, there exists a continuous linear operator which maps l_1 onto X. Now T is strictly singular by the above theorem. However, by [1, Theorem 4, p. 148], T' has a bounded inverse. Thus T' is not strictly singular.

Remark. Any bounded linear operator between Banach spaces whose range does not contain any infinite dimensional closed subspace is clearly strictly singular (such operators were considered by R. S. Phillips in his unpublished note). The above example shows that a strictly singular operator need not have this property. The Kato and Phillips concepts coincide when the domain space X is l_2. Thus whenever the bounded operators from $l_2 \rightarrow Y$ are strictly singular, they have the Phillips property as well.

The following example shows that a strictly singular operator can have a nonseparable range in contrast to the well-known fact that a compact operator always has a separable range.

Example. If Q is an arbitrary set, $l_p(Q)$, $1<p<\infty$, is the space of scalar valued functions x with domain Q, having at most countably many nonzero coordinates, and such that $\|x\| = (\sum_{q\in Q} |x(q)|^p)^{1/p}$ is finite. It is a Banach space with this norm. We assert that all the continuous operators $T: l_2(Q)\to l_p(Q)$, $2<p<\infty$, where Q is an uncountable set, are strictly singular and that the inclusion map is such an operator and has a nonseparable range.

(1) To see that all the operators are strictly singular, suppose some T is not. Then there are closed infinite dimensional subspaces $M\subset l_2(Q)$ and $N\subset l_p(Q)$ such that T restricted to M is an isomorphism between M and N. Choose a countable linearly independent subset $\{x_n\}$ of M. Then T is an isomorphism between $sp\{x_n\}$, the set spanned by $\{x_n\}$, and $sp\{Tx_n\}$. Let C be $\{q\in Q: x_n(q)\neq 0$, some $n\}$. The set C is countable therefore $R=\{x\in l_2(Q): x(q)=0,\ q\notin C\}$ is isometrically isomorphic with l_2. Let $D=\{q\in Q: (Tx)(q)\neq 0$, some $x\in R\}$. The set is countable so the set $S=\{y\in l_p(Q): y(q)=0,\ q\notin D\}$ is isometric to $l_p(Q)$. Since $R(T)\subset S$, the restriction of T to R, T_R, can be regarded as a continuous operator from l_2 to l_p. When restricted to $[sp]^-\{x_n\}$, T_R is an isometry between $[sp]^-\{x_n\}$ and $[sp]^-\{Tx_n\}$, and they are closed infinite dimensional subspaces of R and S respectively, thus T_R is not strictly singular. But this contradicts the first part of the above theorem.

(2) Jensen's inequality shows, just as in the case of $T: l_2\to l_p$ in the above theorem, that the inclusion map I is continuous. To see that the range of I is not separable, note that the elements of $l_p(Q)$ which are characteristic functions of single points are in the range of I. They are mutually separated by distance $2^{1/p}$ and are uncountable.

Remark. Arguments similar to that used in (1) show that both parts of the (a) part of the above theorem remain valid when l_2 and l_p are replaced by $l_2(Q_1)$ and $l_p(Q_2)$ respectively. From the fact that l_1 and c_0 contain no infinite dimensional reflexive subspaces, it follows from similar arguments that $l_1(Q)$ and $c_0(Q)$ do not contain any infinite dimensional reflexive subspaces.

References

1. S. Banach, *Operations linéaires*, Chelsea, New York, 1955.

2. S. Banach and S. Mazur, *Zur Theorie der linearen Dimension*, Studia Math. **4** (1933), 100–112.

3. T. Kato, *Perturbation theory for nullity, deficiency and other quantities of linear operators*, J. Analyse Math. **6** (1958), 261–321.

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