

ORDER PROPERTIES OF BOUNDED OBSERVABLES

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Continuing the development [4] of an aspect of the approach to the axiomatization of quantum mechanics of G. W. Mackey [3], we consider here the real linear space X of signed measures on the set P of events generated by the states, and the set Y_0 of linear functionals on X which are induced in a natural way by the bounded observables. A necessary and sufficient condition for two events to be simultaneously measurable is found in terms of the order structure of Y_0 , with the following consequence: if Y_0 is a lattice, P is deterministic. At the opposite extreme, Y_0 is said to be an "anti-lattice"¹ if the greatest lower bound exists only for comparable pairs of its elements and we show in this case that the center of P is trivial.

Our results extend those of R. V. Kadison [1], in which Y_0 and P are the self-adjoint operators and projections respectively in a uniformly closed self-adjoint operator algebra. While the framework and plan of the proofs were inspired by Kadison's work, almost none of the apparatus used by him is available here with the result that, in detail, our techniques are quite different from his.

Let P be a weakly modular partially ordered set (see [4]). A function x from P to the non-negative real numbers and $+\infty$ is said to be a *measure* if $x(0)=0$ and x is countably additive in the sense that whenever $\{a_i\}$ is a pairwise orthogonal sequence of elements of P , then $x(\bigcup a_i) = \sum x(a_i)$. If x is a measure and $\{b_i\} \subset P$ is an increasing (decreasing) sequence with supremum (infimum) b , then $x(b_i) \rightarrow x(b)$. A countably additive function x from P to the extended real numbers is a *signed measure* if $x(0)=0$ and x takes on at most one of the values $+\infty$ and $-\infty$; x is *finite* if $x(1)$ is finite. Define the functions s and i on the signed measures on P by $s(x) = \sup\{x(a) : a \in P\}$, $i(x) = \inf\{x(a) : a \in P\}$ and set $\|x\| = s(x) - i(x)$. Clearly $\|x\| < \infty$ if and only if x is finite. It is easy to see that $\|x\| = \sup\{x(a) - x(a') : a \in P\}$.

LEMMA 1. *Let X be a real linear space of finite signed measures on P . Then the function $\|\cdot\|$ defined above is a norm for X and, under this norm and its natural partial ordering, X is a partially ordered normed linear space. That is,*²

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¹ Kadison [1].

² See [2].

- (1) $x \geq y$ and $y \geq x$ imply $x = y$.
- (2) $x \geq y$ and $y \geq z$ imply $x \geq z$.
- (3) $x \geq y$ and $\lambda \geq 0$ imply $\lambda x \geq \lambda y$.
- (4) $x \geq y$ implies $x + z \geq y + z$ for all z .
- (5) $x_n \geq 0$, $\|x_n - x\| \rightarrow 0$ imply $x \geq 0$.

In addition

- (6) $x \geq 0$ and $y \geq 0$ imply $\|x + y\| = \|x\| + \|y\|$.

PROOF. Since $i(x) \leq s(x)$, $\|x\| \geq 0$; clearly, $\|x\| = 0$ if and only if $x = 0$. If $\lambda \geq 0$, $\|\lambda x\| = \lambda \|x\|$ is obvious while $\|x\| = \|-x\|$ follows from the fact that $s(-x) = -i(x)$. Finally, $s(x+y) \leq s(x) + s(y)$ and $i(x+y) \geq i(x) + i(y)$ so $\|x+y\| \leq \|x\| + \|y\|$, and $\|\cdot\|$ is a norm for X .

Properties (1)-(4) are obvious. If $x(a) < 0$ for some $a \in P$, $\|x_n - x\| \geq s(x_n - x) \geq -x(a)$ since $x_n \geq 0$. This contradiction of the hypothesis $\|x_n - x\| \rightarrow 0$ proves (5), while (6) follows at once from the fact that $\|z\| = z(1)$ for $z \geq 0$.

A function A from the set \mathcal{B} of Borel subsets of the real line R to P is said to be an *observable* if it has the following properties:

- (7) $A_\emptyset = 0$, $A_R = 1$.
- (8) If E_1, E_2, \dots are pairwise disjoint Borel sets, then A_{E_1}, A_{E_2}, \dots are pairwise orthogonal members of P and $A_{\cup E_i} = \vee A_{E_i}$.

The spectrum of an observable A is defined as follows. Let $N = \{\text{open } E \subset R: A_E = 0\}$ and choose $\{E_i\} \subset N$ with $\cup E_i = \cup_{E \in N} E$ and $E_1 \subset E_2 \subset \dots$. Then $A_{\cup E_i} = \vee A_{E_i} = 0$ so $\cup E_i$ is the largest member of N and we set $\text{spectrum } A = \text{complement } \cup E_i$. Define the *norm* of A , $\|A\|$, as $\sup\{|\lambda|: \lambda \in \text{spectrum } A\}$ and say A is *bounded* if $\|A\| < \infty$. Let O , respectively O_b , denote the set of all, respectively all bounded, observables.

Every signed measure x on P determines a function m_x from O to the signed measures on \mathcal{B} : $m_x(A)(E) = x(A_E)$. If x is a state then $m_x(A)$ is clearly a probability measure on \mathcal{B} for every observable A . It is easy to see that $\|x\| = \sup\{\|m_x(A)\|: A \in O\}$.

Let X be a real linear space of finite signed measures on P . Each observable A , respectively element a of P , defines a linear functional L_A , respectively L_a , on X by $L_A(x) = \int_{-\infty}^{\infty} \lambda dm_x(A)(\lambda)$, respectively $L_a(x) = x(a)$. Evidently, if A is the observable with $A_{\{1\}} = a$ and $A_{\{0\}} = a'$, then $L_A = L_a$.

LEMMA 2. Let x be a finite signed measure on P . Then $\|x\| = \sup\{|L_A(x)|: \|A\| = 1\}$.

PROOF. Let $\lambda = \sup\{|L_A(x)|: \|A\| = 1\}$. Clearly $|L_A(x)| \leq \|x\| \|A\|$ holds for all observables A so $\lambda \leq \|x\|$. On the other hand, we may

choose $a_n \in P$ so that $x(a_n) - x(a'_n) \rightarrow \|x\|$ and let $A(n)$ denote the observable with $A(n)_{\{1\}} = a_n$, $A(n)_{\{-1\}} = a'_n$. Then $\|A(n)\| = 1$, $L_{A(n)}(x) \rightarrow \|x\|$ and the result follows.

COROLLARY. *If A is a bounded observable, L_A is a bounded linear functional on X and $\|L_A\| \leq \|A\|$.*

Now suppose P is the set of events of a system S , P of states and events,³ and henceforward let X denote the real linear space of signed measures on P obtained as the set of all finite real linear combinations of members of S .

LEMMA 3. *For $A \in O_b$, $\|L_A\| = A$.*

PROOF. Since spectrum A is a closed subset of R , we may choose $\lambda \in$ spectrum A with $|\lambda| = \|A\|$. Then if $\epsilon > 0$, it follows from the definition of spectrum A that the event $a = A_{(\lambda-\epsilon, \lambda+\epsilon)} \neq 0$ and hence there exists $f \in S$ with $f(a) = 1$. Then $\lambda - \epsilon \leq |L_A(f)| \leq \lambda + \epsilon$ and, since $\|f\| = 1$, $\|L_A\| = \sup \{ |L_A(x)| : x \in X, \|x\| = 1 \} \geq |L_A(f)| \geq \|A\| - \epsilon$. Thus, $\|L_A\| \geq \|A\|$, and the opposite inequality is supplied by the preceding corollary.

Let $Y_0 = \{L_A : A \in O_b\}$, let Y_0^+ denote the set of non-negative members of Y_0 and let Y denote the subspace of X^* generated by Y_0 . Note that the mapping $a \rightarrow L_a$ is an isomorphism of the partially ordered set P in the partially ordered set Y_0 .

LEMMA 4. *Let T be a subset of Y_0 containing 0, L_a , L_b and L_{ab} for some $a, b \in P$. Then if L_a and L_b have a greatest lower bound L_A in T , $L_A = L_{ab}$.*

PROOF. $0 \leq L_a, L_b$ implies $0 \leq L_A$ and $\|A\| = \sup \{ L_A(f) : f \in S \} \leq \sup \{ L_a(f) : f \in S \} \leq 1$. Now $f(a) = 0$ implies $m_f(A)$ is concentrated in 0, i.e., $f(A_{\{0\}}) = 1$. It follows that $A'_{\{0\}} \leq a$; similarly $A'_{\{0\}} \leq b$ so $A'_{\{0\}} \leq ab$. Then if $g \in S$, $L_A(g) = \int_0^1 \lambda dm_g(A)(\lambda) \leq g(A_{(0,1]}) = g(A'_{\{0\}}) \leq g(ab) = L_{ab}(g)$ so $L_A \leq L_{ab}$. Making use now of the assumption that A is the greatest lower bound of L_a and L_b we obtain the result and

COROLLARY. $L_{ab} = L_a \wedge L_b$ in Y_0^+ .

THEOREM 1. *The events a and b are simultaneously measurable if and only if there exists a linear subspace W of Y containing $L_a, L_b, L_{a \vee b}$ and L_{ab} such that $L_a \wedge L_b$ exists in $W \cap Y_0$.*

PROOF. Suppose $L_a \wedge L_b = L_A \in W$ as in the statement. Then

³ See [4]; actually, the following weaker set of postulates suffices here: E1, E2, E3, E5, S2 and the following: if $f(a) = 1$ whenever $f(b) = 1$, then $b \leq a$.

$L_A = L_{ab}$ by Lemma 4. Let $y = L_a - L_{ab} = L_{a(ab)'}$, $z = L_b - L_{ab} = L_{b(ab)'}$. If $w \in W$ with $w \leq y, z$ then $w + L_{ab} \leq L_a, L_b$ so $w + L_{ab} \leq L_{ab}$ and $w \leq 0$. It follows that $y \wedge z = 0$ in W . Let $u = L_{a \vee b} - z = L_{(a \vee b)(b' \vee ab)}$; then $y = L_{a(ab)'}$, $\leq L_{a \vee b} = u + z$ so $y - u \leq z$. Since $0 \leq u$, $y - u \leq y$ and hence $y - u \leq y \wedge z = 0$, i.e., $y \leq u$. Thus, $a(ab)' \leq (a \vee b)(b' \vee ab) \leq b' \vee ab$. Joining $b' \vee ab$ to both ends of the inequality gives $a(ab)' \vee ab \vee b' = a \vee b' \leq b' \vee ab$. Since the opposite inequality is trivial, we have, taking orthocomplements, $ba' = b(ab)'$. Hence $b = b(ab)' \vee ba = ba' \vee ba$ and the result follows from [4, Lemma 1.4].

Conversely, if a and b are simultaneously measurable, the subspace W of Y generated by the mutually exclusive events $ab, ab', a'b$ and $a'b'$ is clearly a lattice contained in Y_0 .

COROLLARY. *If Y_0 is a lattice, then P is deterministic.*

PROOF. Suppose Y_0 is a lattice, let a and b be events and let W be the subspace of Y generated by $L_a, L_b, L_{a \vee b}$ and L_{ab} . Then $L_{ab} = L_a \wedge L_b$ in Y_0 by Lemma 4 and the hypothesis so, a fortiori, $L_{ab} = L_a \wedge L_b$ in $W \cap Y_0$. Hence a and b are simultaneously measurable by the theorem and P is deterministic.

If $A \in \mathcal{O}$ and α is a Borel function from R to R , the observable C such that $C_E = A_{\alpha(E)}^{-1}$ for all Borel sets E is clearly unique and is denoted $\alpha(A)$. Observables A and B are *simultaneously measurable* or *commute* if there exist an observable C and Borel functions α and β from R to R such that $A = \alpha(C)$ and $B = \beta(C)$.

LEMMA 5. *Let A and B be commuting bounded observables. Then $L_A \leq L_B$ if and only if whenever α, β and C are two Borel functions and a bounded observable respectively such that $A = \alpha(C)$ and $B = \beta(C)$, $\alpha(\lambda) \leq \beta(\lambda)$ holds for almost all λ relative to $m_f(C)$ for all $f \in S$.*

PROOF. Suppose there exists $f \in S$ and a Borel set E of positive $m_f(C)$ measure such that $\beta(\lambda) < \alpha(\lambda)$ on E . Since $m_f(C)(E) = f(C_E) > 0$, $C_E \neq 0$ so $g(C_E) = 1$ for some $g \in S$. But then $L_B(g) = \int_E \beta(\lambda) dm_g(C)(\lambda) < \int_E \alpha(\lambda) dm_g(C)(\lambda) = L_A(g)$, which proves the nontrivial half of the lemma.

COROLLARY. *Let A and B be commuting bounded observables and suppose that $L_A \leq L_B$. Then $\int_E \lambda dm_f(A)(\lambda) \leq \int_E \lambda dm_f(B)(\lambda)$ for all $f \in S$ and Borel sets E .*

PROOF. Suppose, on the contrary, that there exist f, E such that $\int_E \lambda dm_f(A)(\lambda) > \int_E \lambda dm_f(B)(\lambda)$. But then if α, β and C are as in Lemma 5, $\int_E \alpha(\lambda) dm_f(C)(\lambda) > \int_E \beta(\lambda) dm_f(C)(\lambda)$ and so $\alpha(\lambda) > \beta(\lambda)$ must hold on some Borel subset of E of positive $m_f(C)$ measure.

LEMMA 6. Let A and B be commuting bounded observables and let $W = \{L_C: C \in O_b, C \text{ commutes with } A \text{ and } B\}$. Then L_A and L_B have a supremum and an infimum in W .

PROOF. Let α and β be Borel functions and $C \in O_b$ such that $A = \alpha(C)$, $B = \beta(C)$ and let $D = (\sup\{\alpha, \beta\})(C)$. Obviously, $D \in W$, and $L_D \geq L_A, L_B$ by Lemma 5. Now let D_1 be any member of W such that $L_{D_1} \geq L_A, L_B$, let $E_\alpha = \{\lambda: \alpha(\lambda) \geq \beta(\lambda)\}$ and let $E_\beta = \{\lambda: \alpha(\lambda) < \beta(\lambda)\}$. Then, by Lemma 5, Corollary,

$$\begin{aligned} \int_{E_\alpha} \lambda dm_f(D_1)(\lambda) &\geq \int_{E_\alpha} \lambda dm_f(A)(\lambda) \text{ and} \\ \int_{E_\beta} \lambda dm_f(D_1)(\lambda) &\geq \int_{E_\beta} \lambda dm_f(B)(\lambda) \quad \text{for all } f \in S. \end{aligned}$$

Hence

$$\begin{aligned} L_{D_1}(f) &= \int \lambda dm_f(D_1)(\lambda) \geq \int_{E_\alpha} \lambda dm_f(A)(\lambda) + \int_{E_\beta} \lambda dm_f(B)(\lambda) \\ &= \int_{E_\alpha} \alpha(\lambda) dm_f(C)(\lambda) + \int_{E_\beta} \beta(\lambda) dm_f(C)(\lambda) \\ &= \int \sup\{\alpha, \beta\}(\lambda) dm_f(C)(\lambda) = L_D(\lambda) \end{aligned}$$

and so $L_D = L_A \vee L_B$ in W . Similarly, $L_{(\inf\{\alpha, \beta\})(C)} = L_A \wedge L_B$ in W .

LEMMA 7. Suppose a belongs to the center \mathcal{C} of P and define the observable A by $A_{\{1\}} = a$, $A_{\{0\}} = a'$. Then A belongs to the center of O_b .

PROOF. Let $B \in O_b$ and define a function C from the Borel subsets of the real line to P as follows: $C_E = aB_E \vee a'B_{E-3||B||}$. Clearly $C\phi = 0$ and $C_R = 1$. If E and F are disjoint Borel sets, $aB_E \perp aB_F$ since $B_E \perp B_F$ while $aB_E \perp a'B_{F-3||B||}$ since $a \perp a'$; similarly, $a'B_{E-3||B||}$ is orthogonal to aB_F and $aB_{F-3||B||}$ so $C_E \perp C_F$. If E_1, E_2, \dots are pairwise disjoint, then

$$\begin{aligned} \vee C_{E_i} &= \vee (aB_{E_i} \vee a'B_{E_i-3||B||}) = \vee aB_{E_i} \vee \vee a'B_{E_i-3||B||} \\ &= a \vee B_{E_i} \vee a' \vee B_{E_i-3||B||} = aB_{\cup E_i} \vee a'B_{\cup (E_i-3||B||)} \\ &= aB_{\cup E_i} \vee a'B_{\cup E_i-3||B||} = C_{\cup E_i} \end{aligned}$$

since $a \in \mathcal{C}$ and $B \in O_b$ and it follows that $C \in O_b$. Let α be the characteristic function of the set $-\|B\| \leq \lambda \leq \|B\|$ and let β be a Borel function on the line such that $\beta(\lambda) = \lambda$ for $-\|B\| \leq \lambda \leq \|B\|$ and $\beta(\lambda)$

$=\lambda-3\|B\|$ for $2\|B\|\leq\lambda\leq4\|B\|$. Then if $E\subset[-\|B\|,\|B\|]$, $(\beta(C))_E = C_{\beta^{-1}E} = aB_{\beta^{-1}E} \vee a'B_{\beta^{-1}E-3\|B\|} = aB_E \vee a'B_E = B_E$, i.e., $\beta(C) = B$. Evidently, $\alpha(C) = A$ and so A and B commute; since $B \in O_b$ was arbitrary, $A \in \text{center } O_b$ as was to be proved.

LEMMA 8. *Suppose A and B are commuting observables and γ and δ are Borel functions on the line. Then $\gamma(A)$ and $\delta(B)$ commute.*

PROOF. If $A = \alpha(C)$ and $B = \beta(C)$ then $\gamma(A) = \gamma \circ \alpha(C)$ and $\delta(B) = \delta \circ \beta(C)$.

A partially ordered set is said to be an *anti-lattice* if suprema and infima exist only for comparable pairs of its elements. The Corollary of Theorem 1 asserts that if Y_0 is a lattice, $\mathfrak{C} = P$. The following theorem provides the corresponding conclusion for the opposite extreme.

THEOREM 2. *If Y_0 is an anti-lattice, then $\mathfrak{C} = \{0, 1\}$.*

PROOF. Suppose, on the contrary, that there exists $a \in \mathfrak{C}$ with $0 < a < 1$ and let A and I be the observables defined as follows: $A_{\{1\}} = a$, $A_{\{0\}} = a'$, $I_{\{1\}} = 1$. Then A and I belong to the center of O_b by Lemma 7 and $2A \in \text{center } O_b$ also by Lemma 8. Hence L_{2A} and L_I have a supremum and an infimum in Y_0 by Lemma 6 and it follows from the hypothesis that L_{2A} and L_I are comparable. But by S2 we can find f, g in S such that $f(a) = 1$ and $g(a) = 0$. Then $L_{2A}(f) = 2 > 1 = L_I(f)$ while $L_{2A}(g) = 0 < 1 = L_I(g)$ and this contradiction shows that $0 < a < 1$ is incompatible with $A \in \mathfrak{C}$.

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