THE SEMI-GROUP PROPERTY OF POISSON TRANSFORMATION AND SNOW'S INVERSION FORMULA

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To Einar Hille, on the occasion of his retirement from
Yale University, June, 1962.

1. **The Poisson transform.** By the set F we shall mean the family
of functions f which are integrable on each finite interval of the real
axis and for which the integrals

$$(1) \qquad \int^{\infty} \frac{f(x)}{x^2}\, dx \quad \text{and} \quad \int_{-\infty} \frac{f(x)}{x^2}\, dx$$

converge. For each member f of this set and each positive number a
the Poisson transform $P_a f$ is defined by the point function

$$(2) \qquad (P_a f)(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{af(\xi)}{(x-\xi)^2 + a^2}\, d\xi, \qquad -\infty < x < \infty,$$

which under condition (1) converges uniformly in each interval
$0 < a_1 \leqq a \leqq a_2$, so that for fixed x, $(P_a f)(x)$ is continuous in $a > 0$.

The semi-group property, easy to prove in case the integrals (1)
converge *absolutely*, states that

$$(3) \qquad P_b P_a f = P_{a+b} f, \qquad a > 0,\ b > 0.$$

The main purpose of this paper is to establish the validity of this
property for each f in F, thus removing the requirement of absolute
convergence.

As an application it will be established in the last section of the
paper that Snow's inversion formula for the integral transform

$$(4) \qquad g(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{(x-\xi)^2 + 1}\, d\xi$$

holds on the sole assumption that the integrals (1) converge. His own
conditions are more restrictive (for a discussion and reference see
[1, p. 94]).

2. **Proof of the semi-group property.** Let us denote by $k(x, a)$ the
function

$$k(x, a) = \frac{1}{\pi} \frac{a}{x^2 + a^2}.$$

The formula (2) states that

$$\int_{-\infty}^{\infty} k(x - \xi, b)d\xi \int_{-\infty}^{\infty} k(\xi - y, a)f(y)dy = \int_{-\infty}^{\infty} k(x - y, a + b)f(y)dy,$$

$$a > 0, b > 0.$$

In view of the familiar fact that $k(x, a) * k(x, b) = k(x, a+b)$, the proof amounts to showing that

$$\lim_{R,S \to \infty} \frac{1}{\pi^2} \int_{-S}^{R} \frac{bd\xi}{(x - \xi)^2 + b^2} \int_{-\infty}^{\infty} \frac{af(y)dy}{(\xi - y)^2 + a^2}$$

$$= \frac{1}{\pi^2} \int_{-\infty}^{\infty} f(y)dy \int_{-\infty}^{\infty} \frac{abd\xi}{[(x - \xi)^2 + b^2][(\xi - y)^2 + a^2]}.$$

The inside integral on the left-hand side converges uniformly on finite intervals $-S \leq \xi \leq R$. This permits the interchange of the iterated integrals on the left-hand side, so we must show that the difference between

$$\int_{-\infty}^{\infty} f(y)dy \int_{-S}^{R} \cdots d\xi$$

and the right-hand side approaches zero as $R \to \infty$, $S \to \infty$. By $\cdots$ we mean

$$\frac{ab}{[(x - \xi)^2 + b^2][(\xi - y)^2 + a^2]}.$$

This difference can be written

$$\int_{-\infty}^{\infty} f(y)dy \int_{R}^{\infty} \cdots d\xi + \int_{-\infty}^{\infty} f(y)dy \int_{-\infty}^{-S} \cdots d\xi,$$

where the convergence of each of the double integrals follows from subsequent considerations.

We shall show that as $R \to \infty$ the first of these iterated integrals vanishes; the proof that the second does as $S \to \infty$ is similar.

Let $\Delta(x, y)$ denote the expression defined by

(5)
$$\Delta \equiv (x - y)^4 + 2(b^2 + a^2)(x - y)^2 + (b^2 - a^2)^2$$
$$= [(x - y)^2 + (b + a)^2][(x - y)^2 + (b - a)^2].$$

Then by standard methods of integration

$$\int_{-\infty}^{\infty} f(y)dy \int_{R}^{\infty} \cdots d\xi$$

(6)
$$= \int_{-\infty}^{\infty} f(y) \left(\frac{\pi}{2} - \arctan \frac{R-x}{b} \right) \frac{a}{\Delta} [(x-y)^2 - b^2 + a^2] dy$$

$$+ \int_{-\infty}^{\infty} f(y) \left(\frac{\pi}{2} - \arctan \frac{R-y}{a} \right) \frac{b}{\Delta} [(x-y)^2 + b^2 - a^2] dy$$

$$- \int_{-\infty}^{\infty} f(y) \frac{ab}{\Delta} (y-x) \log \frac{(R-x)^2 + b^2}{(R-y)^2 + a^2} dy.$$

The independent convergence of each of these three integrals on the right follows from the convergence of the integrals (1).

As $R \to \infty$ the first of these three approaches zero since the factor $\pi/2 - \arctan((R-x)/b)$ can be moved outside the integral.

The second integral can be written

$$I \equiv \int_{-\infty}^{\infty} \left(\frac{\pi}{2} - \arctan \frac{R-y}{a} \right) dG(y),$$

where

$$G(y) = - \int_{y}^{\infty} \frac{b}{\Delta} [(x-\eta)^2 + b^2 - a^2] f(\eta) d\eta, \qquad \Delta = \Delta(x, \eta).$$

On integrating by parts we find that the integrated parts vanish, leaving

$$I = \int_{-\infty}^{\infty} G(y) \frac{a}{(R-y)^2 + a^2} dy.$$

Consequently,

$$I = \int_{-\infty}^{\infty} G(R-u) \frac{a}{u^2 + a^2} du.$$

This vanishes as $R \to \infty$ because $G(\infty) = 0$ and G is a bounded function.

The third integral on the right-hand side of (6) can be written as

$$J \equiv \int_{-\infty}^{\infty} \frac{y-x}{(x-y)^2 + (b-a)^2} \log \frac{(R-x)^2 + b^2}{(R-y)^2 + a^2} d F(y),$$

where

$$F(y) = - \int_{-\infty}^{y} \frac{abf(\eta)d\eta}{(x-\eta)^2 + (a+b)^2}.$$

This follows from (5). Integration by parts yields

$$J \equiv \int_{-\infty}^{\infty} F(y) \frac{\partial}{\partial y} \left\{ \frac{y-x}{(x-y)^2 + (b-a)^2} \log \frac{(R-x)^2 + b^2}{(R-y)^2 + a^2} \right\} dy$$

$$(7) \qquad = 2 \int_{-\infty}^{\infty} F(y) \frac{y-x}{(x-y)^2 + (b-a)^2} \frac{R-y}{(R-y)^2 + a^2} dy$$

$$+ \int_{-\infty}^{\infty} F(y) \frac{(b-a)^2 - (y-x)^2}{[(x-y)^2 + (b-a)^2]^2} \log \frac{(R-x)^2 + b^2}{(R-y)^2 + a^2} dy.$$

The function $F(y)$ is bounded. Consequently the first integral on the right-hand side of (7) is bounded by a constant independent of R times

$$(8) \qquad \int_{-\infty}^{\infty} \frac{1}{1+|y|} \frac{1}{1+|R-y|} dy.$$

This can be computed to show that it vanishes as $R \to \infty$.

As for the last integral, it is dominated by a constant independent of R times

$$\int_{-\infty}^{\infty} \frac{1}{1+(x-y)^2} \left| \log \frac{(R-x)^2 + b^2}{(R-y)^2 + a^2} \right| dy.$$

Let $y - x = \eta$, $R - x = T$. It remains only to show

$$\lim_{T \to \infty} \int_{-\infty}^{\infty} \frac{1}{1+\eta^2} \left| \log \frac{T^2 + b^2}{(T-\eta)^2 + a^2} \right| d\eta = 0.$$

The integral here is

$$(9) \qquad \begin{aligned} &\leqq \int_{-\infty}^{\infty} \frac{1}{1+\eta^2} \left| \log \frac{T^2 + a^2}{(T-\eta)^2 + a^2} \right| d\eta \\ &+ \int_{-\infty}^{\infty} \frac{1}{1+\eta^2} \left| \log \frac{T^2 + b^2}{T^2 + a^2} \right| d\eta \end{aligned}$$

of which the latter clearly vanishes as $T \to \infty$.

The first integral of (9) can be written

$$\int_{-\infty}^{\infty} \left| \log \frac{T^2 + a^2}{(T-\eta)^2 + a^2} \right| dG(\eta),$$

where $G(\eta) = \arctan \eta - (\pi/2) \operatorname{sgn} \eta$. Integration by parts shows that this integral is numerically less than

$$2 \int_{-\infty}^{\infty} |G(\eta)| \frac{|T-\eta|}{(T-\eta)^2 + a^2} d\eta.$$

Since $|G(\eta)| \leqq A/(1+|\eta|)$ this integral is less than a constant (independent of T), times

$$\int_{-\infty}^{\infty} \frac{1}{1+|\eta|} \frac{1}{1+|T-\eta|} d\eta.$$

This vanishes as $T \to \infty$ for the reason given for (8).

The proof of (3) is now complete.

3. **Snow's inversion formula for the transform** (4). On the assumption that the integrals (1) converge, the integral (4) converges for $-\infty < x < \infty$ and admits analytic continuation into the region $|Iz| < 1$, where $z = x+iy$ [2, p. 543]. Under severe restrictions Snow [1] has shown that (4) is inverted by

$$f(x) = \lim_{y \to 1} [g(x+iy) + g(x-iy)] - (P_1 g)(x).$$

We show that *this holds for almost all x under the sole assumption that the integrals* (1) converge. First by the continuity of $(P_y f)(x)$ as a function of y the formula can be written as

$$(10) \qquad f(x) = \lim_{y \to 1} [g(x+iy) + g(x-iy) - (P_y g)(x)].$$

Next it follows from (4) that

$$g(x+iy) + g(x-iy)$$
$$(11)$$
$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left[\frac{1}{(x+iy-\xi)^2+1} + \frac{1}{(x-iy-\xi)^2+1} \right] f(\xi)d\xi.$$

Now $g = P_1 f$ so that by the semi-group property

$$P_y g = P_y P_1 f = P_{y+1} f.$$

Hence,

$$(12) \qquad (P_y g)(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(1+y)}{(x-\xi)^2+(y+1)^2} d\xi.$$

Subtract (12) from (11). By the identity

$$\frac{1}{(\omega+iy)^2+1} + \frac{1}{(\omega-iy)^2+1} - \frac{1+y}{\omega^2+(y+1)^2} = \frac{1-y}{\omega^2+(1-y)^2}$$

(with $\omega = x - \zeta$), we obtain

$$g(x+iy) + g(x-iy) - (P_y g)(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\tau}{(x-\xi)^2+\tau^2} f(\xi)d\xi,$$

where $\tau = 1 - y$. Now as $y \to 1$, $\tau \to 0$, and it is known [2] that the last integral approaches $f(x)$ almost everywhere as $\tau \to 0$. This proves (10).

BIBLIOGRAPHY

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ON HYPONORMAL OPERATORS

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A bounded linear operator T on a Hilbert space $\mathfrak{H}$ is said to be *hyponormal* in case $\|T^*x\| \leq \|Tx\|$ for all $x \in \mathfrak{H}$. This short note gives a negative answer to the question raised in [1, p. 188]: "Does there exist a completely continuous hyponormal operator which is not normal?"

THEOREM. *If T is hyponormal, $\|T^n\| = \|T\|^n$ for all n.*

PROOF. It is sufficient to prove that $\|T\| = 1$ implies $\|T^n\| = 1$ for all n. Consider the following property:

(C_n) For every $\epsilon > 0$, there exists a unit vector x such that

$$\|T^n x\| \geq 1 - \epsilon \text{ and } \|T^n x - T^* T^{n+1} x\| \leq \epsilon.$$

(C_0) just says that 1 is an approximate proper value for the self-adjoint operator T^*T (see [1, p. 170]). (C_n) obviously implies $\|T^n\| = 1$. Now suppose that (C_n) is valid. For $\epsilon > 0$ and x (indicated in (C_n))

$$\|T^{n+1}x - T^*T^{n+2}x\|^2$$
$$= \|T^{n+1}x\|^2 - 2\|T^{n+2}x\|^2 + \|T^*T^{n+2}x\|^2$$
$$\leq \|T^n x\|^2 - \|T^{n+2}x\|^2 \text{ (because } \|T\| = \|T^*\| = 1)$$
$$\leq \|T^n x\|^2 - \|T^*T^{n+1}x\|^2 \text{ (because } T \text{ is hyponormal)}$$
$$\leq \|T^n x - T^*T^{n+1}x\|\{\|T^n x\| + \|T^*T^{n+1}x\|\} \leq 2\epsilon \text{ by } (C_n).$$