CONDITIONS FOR A MATRIX TO COMMUTE WITH ITS INTEGRAL

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1. **Introduction.** Let $U(t)$ be an $n\times n$ matrix whose elements are continuous functions of a parameter t. We shall find necessary and sufficient conditions for the relation

$$U(t)\int_0^t U(s)ds = \left(\int_0^t U(s)ds\right)U(t) \tag{1.1}$$

to hold in an interval $0\leqq t\leqq t_0$, where t_0 is so small that throughout the interval $[0, t_0]$ the Jordan canonical form of $U(t)$ has the same form. That is, its off-diagonal elements do not change in the interval.

Matrices $U(t)$ satisfying (1.1) are of interest for various reasons; see, for instance, [**1**, p. 278]. We may mention two occasions where (1.1) occurs. Firstly, consider a system of n homogeneous linear differential equations of the first order for n unknown functions with $U(t)$ as the matrix of coefficients. If we consider the unknown functions as components of a vector, and if we form a matrix Y, the n columns of which are n linearly independent solutions of our system, then we have for $Y=Y(t)$:

$$\dot{Y} = UY, \qquad Y(0) = I, \tag{1.2}$$

where a dot denotes the derivative with respect to t and where I denotes a nonsingular matrix which we may choose to be the unit matrix. If (1.1) holds, then (1.2) can be solved in terms of quadratures. In fact, we have

$$Y = \exp\int_0^t U(s)ds. \tag{1.3}$$

Secondly, consider the following problem in the theory of systems with periodic coefficients. Let $W(t)$ be an $n\times n$ matrix depending continuously on t such that, for a constant ω,

$$W(t+\omega) = W(t), \tag{1.4}$$

and also

$$W(-t) = -W(t). \tag{1.5}$$

Then it has been shown in special cases by Demidovic [**2**] and, more

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generally, by the author [3] that a matrix $Z_0(t)$ satisfying

$$\dot{Z}_0 = WZ_0, \qquad Z_0(0) = I, \tag{1.6}$$

is periodic with period ω, i.e., $Z_0(t+\omega)=Z_0(t)$. The matrices $W(t)$ form a linear space under addition.

We ask whether we can extend this linear space such that a system of the type (1.6) still will have periodic solutions. A partial answer to this question is given by the following remark: Let W be a fixed matrix satisfying (1.4) and (1.5). Let $E(t)$ be such that

$$E(t + \omega) = E(t), \qquad E(-t) = E(t). \tag{1.7}$$

Then the system

$$\dot{Z} = (W + \epsilon E)Z, \qquad Z(0) = I$$

will have solutions with the property

$$Z(t + \omega) = Z(t),$$

for all values of ϵ if the matrix $A(t)$ defined by

$$A(t) = Z_0^{-1}(t)E(t)Z_0(t)$$

commutes with its integral. The proof is based on the standard procedure of expanding $Z(t)$ in a power series in ϵ.

2. **Matrices commuting with their derivatives.** Instead of $U(t)$ we introduce

$$V(t) = \int_0^t U(s)ds,$$

and assume that

$$\dot{V}V - V\dot{V} = 0. \tag{2.1}$$

Consider an interval (t_1, t_2) such that, for $t_1 \leqq t \leqq t_2$, there exists a differentiable nonsingular matrix $P(t)$ such that

$$V(t) = P^{-1}(t)J(t)P(t), \tag{2.2}$$

where $J(t)$ is in Jordan canonical form. This means that

$$J = \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & C_r \end{pmatrix}, \tag{2.3}$$

where the submatrices $C_\rho(t)$, $\rho=1, \cdots, r$, are $n_\rho \times n_\rho$ matrices of the form

$$C_\rho = \alpha_\rho(t)I_\rho + \delta_\rho E_\rho. \tag{2.4}$$

Here $\alpha_\rho(t)$ is a differentiable function of t, I_ρ is the $n_\rho \times n_\rho$ unit matrix, δ_ρ is 0 or 1, and E_ρ is the $n_\rho \times n_\rho$ matrix with elements

$$e_{v,\mu}, \qquad v, \mu = 1, \cdots, n_\rho,$$

and

$$e_{v,v+1} = 1, \qquad e_{v,\mu} = 0 \qquad\qquad \text{for } \mu - v \neq 1. \tag{2.5}$$

We shall assume that the interval (t_1, t_2) is such that no difference $\alpha_\rho - \alpha_\sigma$ vanishes in a subinterval unless it vanishes identically.

We may assume that, if $\alpha_\rho - \alpha_\sigma$ vanishes identically, for $\rho \neq \sigma$, either $\delta_\rho \neq 0$ or $\delta_\sigma \neq 0$. Otherwise, we could contract C_ρ and C_σ into a single diagonal matrix.

Now we have:

THEOREM 1. *The general $n \times n$ matrix $V(t)$ satisfying* (2.1) *and having a Jordan canonical form determined by* (2.3), (2.4) *with constant n_ρ, δ_ρ for $t_1 \leqq t \leqq t_2$ is obtained by finding all $n \times n$ matrices X satisfying*

$$J(XJ - JX) - (XJ - JX)J = 0, \tag{2.6}$$

determining the nonsingular solutions $P(t)$ of the matrix differential equation

$$\dot{P} = XP,$$

and forming

$$V = P^{-1}JP.$$

The matrices X form a linear space (*under addition*) *which depends only on the n_ρ, δ_ρ, and the set of pairs of subscripts* (ρ, σ) *for which $\alpha_\rho - \alpha_\sigma$ vanishes identically.*

PROOF. We observe that, trivially,

$$JJ = J\dot{J}. \tag{2.7}$$

By differentiating (2.2), we find

$$\dot{V} = P^{-1}JP + P^{-1}JP + P^{-1}JP. \tag{2.8}$$

Because of $P^{-1}P = I$ we have

$$\dot{P}^{-1}P + P^{-1}P = 0, \qquad \dot{P}^{-1} = -P^{-1}PP^{-1},$$

and therefore from (2.2), (2.8), with $X = \dot{P}P^{-1}$:

$$\dot{V}V - V\dot{V} = P^{-1}\{-XJ + J + JX\}JP - P^{-1}J\{-XJ + J + JX\}P = 0.$$

If we multiply this last equation by P on the left and P^{-1} on the right and then make use of (2.7) we get (2.6). We note that the solutions X of (2.6) form a linear space. In the next section, we shall determine a basis for the linear space of the matrices X and, incidentally, shall also prove that this space does not depend on the functions $\alpha_\rho(t)$ but merely on the discrete parameters mentioned in Theorem 1.

COROLLARY. *A system of linear differential equations which, in matrix form, can be written as*

$$\dot{Y} = UY \tag{2.9}$$

where the coefficient matrix $U=\dot{V}$ *has the property* $UV=VU$, *can always be transformed into a system*

$$\dot{Z} = (X + \dot{J} + JX - XJ)Z, \tag{2.10}$$

where X, J *are defined as in Theorem* 1. *The transformation to be used is, of course,* $Z=PY$, *where* P *is defined as in Theorem* 1.

3. **The space of matrices** X. The solutions X of (2.6) may be written as matrices which are composed of submatrices $X_{\rho,\sigma}$, $\rho,\sigma=1,\cdots,r$, where $X_{\rho,\sigma}$ is a matrix with n_ρ rows and n_σ columns and

$$X = (X_{\rho,\sigma}) \tag{3.1}$$

with the natural arrangement of the submatrices. From (2.6) we find the equations

$$C_\rho^2 X_{\rho\sigma} + X_{\rho\sigma}C_\sigma^2 - 2C_\rho X_{\rho\sigma}C_\sigma = 0, \tag{3.2}$$

where C_ρ is given by equation (2.4).

If we let x_{kl} denote the element in the kth row and lth column of $X_{\rho\sigma}$ then (3.2) gives us the scalar equations

$$\begin{aligned}(\alpha_\rho - \alpha_\sigma)^2 x_{k,l} + 2\delta_\rho(\alpha_\rho - \alpha_\sigma)x_{k+1,l} + 2\delta_\sigma(\alpha_\sigma - \alpha_\rho)x_{k,l+1}& \\ + \delta_\rho^2 x_{k+2,l} + \delta_\sigma^2 x_{k,l-2} - 2\delta_\rho\delta_\sigma x_{k+1,l-1} &= 0\end{aligned} \tag{3.3}$$

where

$$k = 1, 2, \cdots, n_\rho, \quad l = 1, 2, \cdots, n_\sigma,$$

and where we define $x_{pq}=0$ if $p>n_\rho$ or $q<1$. Equations (3.3) have to be analyzed for various cases. We may summarize the results as follows:

THEOREM 2. *The matrix* $X_{\rho\sigma}$ *has one of the following structures*:

CASE 1. $\alpha_\rho-\alpha_\sigma$ *does not vanish identically* (*and, therefore, not in any subinterval of* (t_1, t_2). *Then* $X_{\rho\sigma}$ *is identically zero.*

CASE 2. $\alpha_\rho - \alpha_\sigma \equiv 0$, $\delta_\rho = 0$, $\delta_\sigma = 1$. *Then the last two columns of* $X_{\rho,\sigma}$ *are arbitrary, but all other elements of* $X_{\rho,\sigma}$ *vanish identically.*

CASE 3. $\alpha_\rho - \alpha_\sigma \equiv 0$, $\delta_\rho = 1$, $\delta_\sigma = 0$. *Then the first two rows of* $X_{\rho,\sigma}$ *are arbitrary but all other elements of* $X_{\rho,\sigma}$ *vanish identically.*

CASE 4. $\alpha_\rho - \alpha_\sigma \equiv 0$, $\delta_\rho = \delta_\sigma = 0$. *Then we may assume* $\rho = \sigma$ (*see remarks before Theorem* 1), *and* $X_{\rho\rho}$ *is arbitrary.*

CASE 5. $\alpha_\rho - \alpha_\sigma \equiv 0$, $\delta_\rho = \delta_\sigma = 1$. *Denoting the elements of* $X_{\rho,\sigma}$ *by* $x_{l,k}$, where $l = 1, \cdots, n_\rho$ *and* $k = 1, \cdots, n_\sigma$, *and if* $n_\rho > n_\sigma$, *then the first two rows of* $X_{\rho,\sigma}$ *are arbitrary and* $X_{\rho\sigma}$ *has the appearance indicated below*:

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15}, & \cdots \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25}, & \cdots \\ 0 & 2x_{21} & 2x_{22} - x_{11} & 2x_{23} - x_{12} & 2x_{24} - x_{13}, & \cdots \\ 0 & 0 & 3x_{21} & 3x_{22} - 2x_{11} & 3x_{23} - 2x_{12}, & \cdots \\ 0 & 0 & 0 & 4x_{21} & 4x_{22} - 3x_{11}, & \cdots \\ 0 & 0 & 0 & 0 & 5x_{21}, & \cdots \\ \cdot\,\cdot & \cdot\,\cdot & \cdot\,\cdot & \cdot\,\cdot & \cdot\,\cdot & \cdots \end{pmatrix}$$

If $n_\rho < n_\sigma$, *the role of rows and columns has to be exchanged, and if* $n_\rho = n_\sigma$, *the* $X_{\rho,\rho}$ *is triangular, but the same shape as above, except that* $x_{21} = 0$.

Only Case 5 requires a more detailed analysis. However, once the explicit form of X stated above is known, it can be verified with a moderate amount of calculations which will be omitted here.

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