# TAUBERIAN CONSTANTS FOR THE ABEL AND CESARO TRANSFORMATIONS ${ }^{1}$ 

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1. Introduction and main results. Let $\left\{s_{n}\right\}\left(s_{n}=a_{0}+a_{1}+\cdots+a_{n}\right)$ be a sequence of real or complex numbers. Denote by $t(x)$ the Abel transform of $\left\{s_{n}\right\}$, that is

$$
t(x) \equiv \sum_{n=0}^{\infty} a_{n} x^{n}
$$

where the power series on the right is supposed convergent in the unit circle. In addition to classical Abelian and Tauberian theorems which give information about one of $\lim _{x \uparrow 1} t(x)$ and $\lim _{n \rightarrow \infty} s_{n}$ when the other exists, it is possible to find estimates of

$$
\lim _{n \rightarrow \infty, x_{n} \rightarrow \infty}\left|t\left(x_{n}\right)-s_{n}\right| \quad \text { or } \lim _{x \uparrow 1, n(x) \rightarrow \infty}\left|t(x)-s_{n(x)}\right|
$$

when neither $\lim t(x)$ nor $\lim s_{n}$ is supposed to exist.
Some estimates investigated before now are of one of the following two forms. The first form is the following: for any fixed number $q>0$ there exists $a$ finite constant $A_{q}$ such that for any sequence $\left\{s_{n}\right\}$ with a bounded sequence $\left\{n a_{n}\right\}$ ( $n \geqq 0$ ) we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty, x \rightarrow 1, n(1-x) \rightarrow \boldsymbol{q}}\left|s_{n}-\sum_{m=0}^{\infty} a_{m} x^{m}\right| \leqq A_{q} \cdot \lim _{n \rightarrow \infty}\left|n a_{n}\right| \tag{1}
\end{equation*}
$$

The second form is the following: for any fixed number $q>0$ there exists a finite constant $B_{q}$ such that for any sequence $\left\{s_{n}\right\}$ with a bounded sequence $\left\{\left(0 a_{0}+1 a_{1}+2 a_{2}+\cdots+n a_{n}\right)(n+1)^{-1}\right\}(n \geqq 0)$ we have

$$
\begin{align*}
\lim _{n \rightarrow \infty, x \rightarrow 1, n(1-x) \rightarrow \boldsymbol{q}} \mid & \left|s_{n}-\sum_{m=0}^{\infty} a_{m} x^{m}\right| \\
& \leqq B_{q} \cdot \limsup _{n \rightarrow \infty}\left|\frac{a_{1}+2 a_{2}+\cdots+n a_{n}}{n+1}\right| . \tag{2}
\end{align*}
$$

While investigating estimates of the form (1) and (2) we have two problems. The first is to show that a finite constant $A_{q}, B_{q}$ satisfying

[^0](1) or (2) exists. The second problem is to obtain the smallest value of the constants which satisfy (1) or (2).

In this paper we shall prove some theorems which include and generalize known results of the form (1) and (2). In the statement of our results $\gamma$ will denote the Euler constant, logarithms will have the base $e,\left\{c_{n}^{(\alpha)}\right\}$ will denote the Cesàro transform of order $\alpha(\alpha>-1)$ of the sequence $\left\{s_{n}\right\}$, that is

$$
\begin{equation*}
c_{n}^{(\alpha)}=\binom{n+\alpha}{n}^{-1} \sum_{m=0}^{n}\binom{n-m+\alpha-1}{n-m} s_{m}, \quad n \geqq 0 \tag{3}
\end{equation*}
$$

$a_{n}^{(\alpha)}$ will denote the Cesàro transform of order $\alpha$ of the sequence $\left\{n a_{n}\right\}(n \geqq 0)$ and the binomial coefficient

$$
\binom{n+\alpha}{n}
$$

for $n=0,1,2, \cdots$, and any real $\alpha$ is defined by

$$
\begin{equation*}
\binom{0+\alpha}{0}=1,\binom{n+\alpha}{n}=\left(1+\frac{\alpha}{1}\right)\left(1+\frac{\alpha}{2}\right) \cdots\left(1+\frac{\alpha}{n}\right) \tag{4}
\end{equation*}
$$

$$
\text { for } n \geqq 1
$$

It is known that for any fixed real $\alpha, \alpha \neq-1,-2,-3, \cdots$, we have

$$
\binom{n+\alpha}{n} \sim \frac{n^{\alpha}}{\Gamma(\alpha+1)}, \quad \text { as } \quad n \rightarrow \infty
$$

One of the main results of this paper is the following theorem.
Theorem 1. Let $\beta$ and $q$ be two real numbers satisfying $q>0,0 \leqq \beta \leqq 1$. Then for any sequence $\left\{s_{n}\right\}$ satisfying $\left|a_{n}^{(\beta)}\right| \leqq K<+\infty$, for $n=0,1$, 2, •. • we have

$$
\begin{equation*}
\limsup _{n \rightarrow \infty, x \rightarrow 1, n(1-x) \rightarrow q}\left|s_{n}-\sum_{m=0}^{\infty} a_{m} x^{m}\right| \leqq A_{q}^{(\beta)} \cdot \lim \sup _{n \rightarrow \infty}\left|a_{n}^{(\beta)}\right| \tag{5}
\end{equation*}
$$

where

$$
\begin{equation*}
A_{q}^{(\beta)}=\gamma+\log q+\frac{2}{\Gamma(\beta+1)} \int_{q}^{\infty}{ }_{v^{\beta} e^{-v}} \log \frac{v}{q} d v . \tag{6}
\end{equation*}
$$

Moreover, the constant $A_{q}^{(\beta)}$ is the best in the following sense. There is a real sequence $\left\{s_{n}\right\}$ such that $0<\lim \sup _{n \rightarrow \infty}\left|a_{n}^{(\beta)}\right|<+\infty$ and the members on both sides of inequality (5) are equal.

The special case $\beta=0$ of Theorem 1 is due to R. P. Agnew [1].

The special case $\beta=1, q=1$ is due to P. Hartman [3].
That some result like Theorem 1 may be true was suggested by the following heuristic consideration. If an inequality of the form (1) is true for some finite constant $A_{q}$ then Abel summability and $n a_{n}$ $=O(1)$ will imply $s_{n}=O(1)$ (more is known, in fact, that $\left\{s_{n}\right\}$ is even convergent). In addition, if an inequality of the form (2) is true for some finite constant $B_{q}$, then Abel summability and $a_{n}^{(1)}$ $=O(1)$ will imply $s_{n}=O(1)$ (in this case it is known that Abel summability of $\left\{s_{n}\right\}$ and $a_{n}^{(1)}=O(1)$ imply the ( $C, \epsilon$ ) summability of $\left\{s_{n}\right\}$, for each $\epsilon>0$, and by the identity $s_{n}-c_{n}^{(1)}=a_{n}^{(1)},\left\{s_{n}\right\}$ is bounded). Now, it is known ([4], Theorem (6.2) and the remark after it) that if $\left\{s_{n}\right\}$ is summable Abel and for some $\alpha \geqq 0, a_{n}^{(\alpha)}=O(1)$ then $\left\{s_{n}\right\}$ is summable $(C, \alpha-1+\epsilon)$, for each $\epsilon>0$, and in particular $\left\{s_{n}\right\}$ is summable $(C,[\alpha])$ which implies in particular $c_{n}{ }^{([\alpha])}=O(1)$. The truth of (1) and (2) for finite constants $A_{q}, B_{q}$ suggested, by the above consideration, that for $0 \leqq \beta \leqq 1$ inequality (5) may be true for a finite constant $A_{q}^{(\beta)}$; and this is proved in Theorem 1.

The above considerations suggest even more. They suggest the following theorem, which, as we shall show, is true.

Theorem 2. Let the three real numbers $\alpha, \beta, q$ satisfy $q>0$ and $-1<\alpha \leqq \beta \leqq \alpha+1$. Then for any sequence $\left\{s_{n}\right\}$ satisfying $\left|a_{n}^{(\beta)}\right| \leqq K<$ $+\infty$, for $n=0,1, \cdots$, we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty, x \rightarrow 1, n(1-x) \rightarrow q}\left|c_{n}^{(\alpha)}-\sum_{m=0}^{\infty} a_{m} x^{m}\right| \leqq C_{q}^{(\alpha, \beta)} \cdot \lim \sup _{n \rightarrow \infty}\left|a_{n}^{(\beta)}\right| \tag{7}
\end{equation*}
$$

where
$C_{q}^{(\alpha, \beta)}=\gamma+\log q-\int_{0}^{1} \frac{1-(1-u)^{\alpha}}{u} d u+\frac{2}{\Gamma(\beta+1)} \int_{q}^{\infty} v^{\beta} e^{-v} \log \frac{v}{q} d v$.
Moreover, the constant $C_{q}^{(\alpha, \beta)}$ is the best in the following sense. There is a real sequence $\left\{s_{n}\right\}$ such that $0<\lim \sup _{n \rightarrow \infty}\left|a_{n}^{(\beta)}\right|<+\infty$ and the members of inequality (7) are equal.

Theorem 1 is the special case $\alpha=0,0 \leqq \beta \leqq 1$, of Theorem 2 .
While proving Theorem 2 we shall obtain the following result too.
Theorem 3. Suppose $\alpha, \beta$ and $q$ are three real numbers satisfying $-1<\alpha \leqq \beta \leqq \alpha+1,0<q$. Then for any sequence $\left\{s_{n}\right\}$ with a bounded sequence $\left\{a_{n}^{(\beta)}\right\}$ we have

$$
\begin{equation*}
\limsup _{n \rightarrow \infty}\left|c_{n}^{(\alpha)}-c_{n}^{(\beta)}\right| \leqq D_{q}^{(\alpha, \beta)} \cdot \limsup _{n \rightarrow \infty}\left|a_{n}^{(\beta)}\right| \tag{8}
\end{equation*}
$$

where

$$
D_{q}^{(\alpha, \beta)}=\int_{0}^{1} \frac{(1-u)^{\alpha}-(1-u)^{\beta}}{u} d u .
$$

Moreover, the constant $D_{\ell}^{(\alpha, \beta)}$ is the best in the following sense. There is a real sequence $\left\{s_{n}\right\}$ such that $0<\lim \sup _{n \rightarrow \infty}\left|a_{n}^{(\beta)}\right|<+\infty$ and the members of inequality (8) are equal.

The special case $\alpha=0, \beta=1$ of Theorem 3 is a special case of a theorem of V. Garten [2].

In §3 we shall give some results concerning relations between limits points of a sequence $\left\{s_{n}\right\}$ and the limit points of its Abel transform $t(x)$.
2. Proof of Theorem 2 and Theorem 3. In the proof of Theorem 2 and Theorem 3 we shall use the following results.

Theorem A. Suppose $\left\{s_{n}\right\}$ is any bounded (real or complex) sequence. Let $\left\{c_{n}(x)\right\}(n=0,1,2, \cdots)$ be a sequence of functions defined for $0<x<+\infty$ and satisfying

$$
\begin{equation*}
\lim _{x \rightarrow \infty} c_{n}(x)=0 \quad \text { for } n=0,1,2, \cdots \tag{9}
\end{equation*}
$$

and

$$
\begin{equation*}
\limsup _{x \rightarrow \infty} \sum_{n=0}^{\infty}\left|c_{n}(x)\right| \equiv M<+\infty . \tag{10}
\end{equation*}
$$

Then we have

$$
\begin{equation*}
\limsup _{x \rightarrow \infty}\left|\sum_{n=0}^{\infty} c_{n}(x) s_{n}\right| \leqq M \cdot \operatorname{lim\operatorname {sup}}\left|s_{n}\right| \tag{11}
\end{equation*}
$$

Moreover, $M$ is the best constant in the following sense. There exists a bounded sequence $\left\{s_{n}\right\}$ (real if all the $c_{n}(x)$ are real) satisfying $0<\lim \sup _{n \rightarrow \infty}\left|s_{n}\right|<+\infty$ and such that the members of inequality (11) are equal.

Theorem A is due to R. P. Agnew [1].
Theorem B. Suppose $\alpha>-1$. Then for any sequence $\left\{s_{n}\right\}$ we have

$$
\begin{equation*}
s_{n}=\sum_{m=0}^{n}\binom{n-m-\alpha-1}{n-m}\binom{m+\alpha}{m} c_{m}^{(\alpha)}, \quad \text { for } n=0,1,2, \cdots, \tag{12}
\end{equation*}
$$

and

$$
\begin{equation*}
n\left\{c_{n}^{(\alpha)}-c_{n-1}^{(\alpha)}\right\}=a_{n}^{(\alpha)}, \quad \text { for } n=1,2, \cdots \tag{13}
\end{equation*}
$$

If $\alpha$ and $\beta$ are two real numbers satisfying $-1<\alpha<\beta$ then for any sequence $\left\{s_{n}\right\}$ we have

$$
\begin{equation*}
c_{n}^{(\alpha)}=\binom{n+\alpha}{n}^{-1} \sum_{m=0}^{n}\binom{n-m+\alpha-\beta-1}{n-m}\binom{m+\beta}{m} c_{m}^{(\beta)}, \tag{14}
\end{equation*}
$$

for $n=0,1, \cdots$.
This theorem is well-known and easily proved.
Theorem C. Suppose $\alpha$ and $\beta$ are two real numbers satisfying $-1<\alpha \leqq \beta$. Then for $0 \leqq x<1$ and $m=1,2,3, \cdots$, we have

$$
\begin{align*}
& \begin{array}{c}
\sum_{p=0}^{\infty}\binom{p+\alpha-\beta-1}{p}\binom{p+m+\alpha}{p+m}^{-1}(p+m)^{-1} x^{p+m+\alpha} \\
\\
=\int_{0}^{x}(x-t)^{\alpha} t^{m-1}(1-t)^{\beta-\alpha} d t \\
\sum_{p=0}^{\infty}\binom{p+\alpha-\beta-1}{p}\binom{p+m+\alpha}{p+m}^{-1}(p+m)^{-1} \\
\quad=\int_{0}^{1}(1-t)^{\beta} t^{m-1} d t=\frac{\Gamma(\beta+1)(m-1)!}{\Gamma(\beta+m+1)} \geqq 0
\end{array} \tag{15}
\end{align*}
$$

and

$$
\begin{align*}
& \sum_{p=1}^{\infty}\binom{p+\alpha-\beta-1}{p}\binom{p+m}{p+\alpha}^{-1}(p+m)^{-1}  \tag{17}\\
&=\int_{0}^{1}\left\{(1-t)^{\beta}-(1-t)^{\alpha}\right\} t^{m-1} d t
\end{align*}
$$

Proof. We have, for $0 \leqq x<1$, by uniform convergence,

$$
\begin{aligned}
\int_{0}^{x} & (x-t)^{\alpha} t^{m-1}(1-t)^{\beta-\alpha} d t \\
& =\int_{0}^{x}(x-t)^{\alpha} \sum_{p=0}^{\infty}\binom{p+\alpha-\beta-1}{p} t^{p+m-1} d t \\
\quad & =\sum_{p=0}^{\infty}\binom{p+\alpha-\beta-1}{p} \int_{0}^{x}(x-t)^{\alpha} t^{p+m-1} d t \\
& =\sum_{p=0}^{\infty}\binom{p+\alpha-\beta-1}{p}\binom{p+m+\alpha}{p+m}^{-1}(p+m)^{-1} x^{p+m+\alpha}
\end{aligned}
$$

This proves (15). By Abel's theorem, since the series in the left side of (16) is convergent, we have

$$
\begin{aligned}
\sum_{p=0}^{\infty}\binom{p+\alpha-\beta-1}{p}\binom{p+m+\alpha}{p+m}^{-1} & (p+m)^{-1} \\
& =\lim _{x \neq 1} \int_{0}^{x}(x-t)^{\alpha} t^{m-1}(1-t)^{\beta-\alpha} d t
\end{aligned}
$$

(and by the substitution $t=x u$ )

$$
=\lim _{x \uparrow 1} x^{m+\alpha} \int_{0}^{1}(1-u)^{\alpha} u^{m-1}(1-x u)^{\beta-\alpha} d u
$$

(and by Beppo Levi's theorem for integration of monotonic sequences)

$$
=\int_{0}^{1}(1-u)^{\beta} u^{m-1} d u
$$

This proves (16). The proof of (17) is now immediate. Q.E.D.
Theorem D. Suppose $\beta$ is a real number satisfying $\beta>-1$. Then for the series $\sum_{n=0}^{\infty} a_{n}$, where, $a_{0}=0$,

$$
a_{n}=\frac{1}{n}\left\{1-\binom{n-\beta-1}{n}\right\} \quad \text { for } n=1,2, \cdots,
$$

we have $a_{0}^{(\beta)}=0$ and $a_{n}^{(\beta)}=1$ for $n=1,2, \cdots$.
The proof follows immediately from (14).
Theorem E. Let $\alpha$ be a real number satisfying $\alpha>-1$. Then the convergent series $\sum_{n=0}^{\infty} a_{n}$ where $a_{0}=0$ and

$$
a_{n}=\frac{1}{n}\binom{n-\alpha-1}{n}, \quad \text { for } n=1,2, \cdots
$$

is summable ( $C,-1+\epsilon$ ) (to its sum) for each $\epsilon>0$; in particular the series is summable ( $C, \alpha$ ) (to its sum).

Proof. Our series is convergent because, for $\alpha=0,1,2, \cdots$, its terms are zero from some place on, and for $\alpha>-1, \alpha \neq 0,1, \cdots$, $a_{k} \sim\{\Gamma(-\alpha)\}^{-1} k^{-(\alpha+2)}$ as $k \rightarrow \infty$. Now, $n a_{n} \rightarrow 0$ as $n \rightarrow \infty$, therefore, by a well-known theorem, our series is summable ( $C,-1+\epsilon$ ) (to its sum) for each $\epsilon>0$. Since $\alpha>-1$ our series is also summable ( $C, \alpha$ ) to its sum. Q.E.D.

Theorem F. Suppose $\alpha$ and $\beta$ are two real numbers satisfying $-1<\alpha \leqq \beta \leqq \alpha+1$. Then for any sequence $\left\{s_{n}\right\}$ we have, for $n=1,2, \cdots$,

$$
\begin{equation*}
c_{n}^{(\alpha)}-c_{n}^{(\beta)}=\sum_{r=1}^{n} b_{n, r} a_{r}^{(\beta)} \tag{18}
\end{equation*}
$$

where

$$
\begin{equation*}
b_{n, r} \geqq 0 \quad \text { for } 1 \leqq r \leqq n \tag{19}
\end{equation*}
$$

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \sum_{r=1}^{n} b_{n, r}=\int_{0}^{1} \frac{(1-u)^{\alpha}-(1-u)^{\beta}}{u} d u \tag{20}
\end{equation*}
$$

$$
b_{n, r}=\binom{r+\beta}{r} \sum_{p=0}^{n-r}\binom{p+\alpha-\beta-1}{p}\binom{p+r+\alpha}{p+r}^{-1}(p+r)^{-1}-\frac{1}{r}
$$

$$
\text { for } r=1,2, \cdots,
$$

and

$$
\begin{equation*}
\lim _{n \rightarrow \infty} b_{n, r}=0 \quad \text { for } r=1,2,3, \cdots \tag{21a}
\end{equation*}
$$

Proof. We have, by (13), for $n=1,2, \cdots$,

$$
c_{n}^{(\alpha)}-c_{n}^{(\beta)}=\sum_{m=1}^{n} \frac{1}{m} a_{m}^{(\alpha)}-\sum_{r=1}^{n} \frac{1}{r} a_{r}^{(\beta)}
$$

(and by (14), since $a_{0}^{(\alpha)}=a_{0}^{(\beta)}=0$ )

$$
=\sum_{m=1}^{n} \frac{1}{m}\binom{m+\alpha}{m}^{-1} \sum_{r=1}^{m}\binom{m-r+\alpha-\beta-1}{m-r}\binom{r+\beta}{r} a_{r}^{(\beta)}-\sum_{r=1}^{n} \frac{1}{r} a_{r}^{(\beta)}
$$

(and changing the order of summation)

This proves (21). Now, for $p>0$,

$$
\binom{p+\alpha-\beta-1}{p}<0
$$

therefore

$$
b_{n, r} \geqq\binom{ r+\beta}{r} \sum_{p=0}^{\infty}\binom{p+\alpha-\beta-1}{p}\binom{p+r+\alpha}{p-r}^{-1}(p+r)^{-1}-\frac{1}{r}
$$

(and by (16))

$$
=\binom{r+\beta}{r} \frac{\Gamma(\beta+1)(r-1)!}{\Gamma(r+\beta+1)}-\frac{1}{r}=\frac{1}{r}-\frac{1}{r}=0 .
$$

$$
\begin{aligned}
& =\sum_{r=1}^{n} a_{r}^{(\beta)}\left\{\binom{r+\beta}{r} \sum_{m=r}^{n}\binom{m-r+\alpha-\beta-1}{m-r} m^{-1}\binom{m+\alpha}{m}^{-1}-\frac{1}{r}\right\} \\
& =\sum_{r=1}^{n} a_{r}^{(\beta)}\left\{\binom{r+\beta}{r} \sum_{p=0}^{n-r}\binom{p+\alpha-\beta-1}{p}\binom{p+r+\alpha}{p+r}^{-1}(p+r)^{-1}-\frac{1}{r}\right\} \text {. }
\end{aligned}
$$

This proves (19) and (21a). Now, by Theorem $D$, we have for the special series $\sum_{n=0}^{\infty} a_{n}$ with $a_{0}=0$,

$$
a_{n}=\frac{1}{n}\left\{1-\binom{n-\beta-1}{n}\right\} \quad \text { for } n>0
$$

since $a_{0}^{(\beta)}=0$ and $a_{n}^{(\beta)}=1$ for $n>0$,

$$
\begin{aligned}
\sum_{r=1}^{n} b_{n, r} & =c_{n}^{(\alpha)}-c_{n}^{(\beta)} \\
& =\sum_{m=1}^{n} \frac{1}{m} a_{m}^{(\alpha)}-\sum_{m=1}^{n} \frac{1}{m} a_{m}^{(\beta)}
\end{aligned}
$$

(and since

$$
\begin{aligned}
a_{n}= & \frac{1}{n}\left\{1-\binom{n-\alpha-1}{n}\right\}+\left\{\frac{1}{n}\binom{n-\alpha-1}{n}-\frac{1}{n}\binom{n-\beta-1}{n}\right\} \\
= & \sum_{m=1}^{n} \frac{1}{m}-c_{n}^{(\alpha)}\left(0+\sum_{k=1}^{\infty}\left\{\frac{1}{k}\binom{k-\alpha-1}{k}-\frac{1}{k}\binom{k-\beta-1}{k}\right\}\right) \\
& -\sum_{m=1}^{n} \frac{1}{m}
\end{aligned}
$$

therefore, by Theorem E, since $\alpha>-1$ )

$$
\rightarrow \sum_{k=1}^{\infty} \frac{1}{k}\binom{k-\alpha-1}{k}-\sum_{k=1}^{\infty} \frac{1}{k}\binom{k-\beta-1}{k}, \quad \text { as } n \rightarrow \infty,
$$

(and by (17) with $\alpha=0$ there and $\beta$ there is either $\alpha$ or $\beta$ here)

$$
\begin{aligned}
& =-\int_{0}^{1} \frac{1-(1-t)^{\alpha}}{t} d t+\int_{0}^{1} \frac{1-(1-t)^{\beta}}{t} d t \\
& =\int_{0}^{1} \frac{(1-t)^{\alpha}-(1-t)^{\beta}}{t} d t
\end{aligned}
$$

This completes the proof of our theorem. Q.E.D.
Theorem G. Suppose $\beta$ and $q$ are two real numbers satisfying $\beta>-1, q>0$. Then for any sequence $\left\{s_{n}\right\}$ we have

$$
\begin{equation*}
c_{n}^{(\beta)}-\sum_{m=0}^{\infty} a_{m} x^{m}=\sum_{m=1}^{n} d_{n, m}(x) a_{m}^{(\beta)}-\sum_{m=n+1}^{\infty} d_{n, m}(x) a_{m}^{(\beta)} \tag{22}
\end{equation*}
$$

where

$$
\begin{equation*}
d_{n, m}(x) \geqq 0 \quad \text { for } m, n=1,2,3, \cdots, \quad 0 \leqq x<1 \tag{23}
\end{equation*}
$$

$$
\begin{equation*}
\lim _{n \rightarrow \infty, x \rightarrow 1, n(1-x) \rightarrow a} d_{n, m}(x)=0 \quad \text { for } m=1,2, \cdots \tag{23a}
\end{equation*}
$$

and

$$
\begin{align*}
& \quad \lim _{n \rightarrow \infty, x \rightarrow 1, n(1-x) \rightarrow q} \sum_{m=1}^{n} d_{n, m}(x) \\
& =\gamma+\log q-\int_{0}^{1} \frac{1-(1-u)^{\beta}}{u} d u+\frac{2}{\Gamma(\alpha+1)} \int_{q}^{\infty} v^{\beta} e^{-v} \log \frac{v}{q} d v . \tag{24}
\end{align*}
$$

Theorem G is a special case of a general theorem proved for the [ $J, f(x)$ ] transformations in [5]. See the proof of Theorem (3.1) and Example (4.4) there.

Proof of Theorem 3. The proof of Theorem 3 follows immediately from Theorem F and Theorem A. Q.E.D.

Proof of Theorem 2. We have for $0 \leqq x<1$

$$
c_{n}^{(\alpha)}-\sum_{m=0}^{\infty} a_{m} x^{m}=c_{n}^{(\alpha)}-c_{n}^{(\beta)}+c_{n}^{(\beta)}-\sum_{m=0}^{\infty} a_{m} x^{m}
$$

(and by Theorem F and Theorem G)

$$
=\sum_{r=1}^{n}\left\{b_{n, r}+d_{n, r}(x)\right\} a_{r}^{(\beta)}-\sum_{r=n+1}^{\infty} d_{n, r}(x) a_{r}^{(\beta)} .
$$

In order to complete the proof it is enough, by Theorem A, to show that

$$
I \equiv \lim _{n \rightarrow \infty, x \rightarrow 1, n(1-x) \rightarrow q}\left|b_{n, r}+d_{n, r}(x)\right|=0 \quad \text { for } r=1,2, \cdots
$$

and

$$
J \equiv \lim _{n \rightarrow \infty, x \rightarrow 1, n(1-x) \rightarrow \ell}\left\{\sum_{r=1}^{n}\left|b_{n, r}+d_{n, r}(x)\right|+\sum_{r=n+1}^{\infty}\left|d_{n, r}(x)\right|\right\}=C_{q}^{(\alpha, \beta)}
$$

By Theorem F and Theorem G we have $I=0$. Now, by (19) and (23),

$$
J=\lim _{n \rightarrow \infty} \sum_{r=1}^{n} b_{n, r}+\lim _{n \rightarrow \infty, x \rightarrow 1, n(1-x) \rightarrow \boldsymbol{q}} \sum_{r=1}^{\infty} d_{n, r}(x)
$$

(and by (20) and (24))

$$
=C_{q}^{(\alpha, \beta)} .
$$

Q.E.D.
3. Conclusion. It is easy to prove the following result for the num-
ber $C_{q}^{(\alpha, \beta)}$ as a function of $q>0$.
Theorem 4. Suppose $\alpha$ and $\beta$ are two real numbers satisfying $-1<\alpha \leqq \beta \leqq \alpha+1$. If for $q>0 C_{q}^{(\alpha, \beta)}$ is defined by (8) then

$$
\begin{align*}
& C_{q}^{(\alpha, \beta)} \geqq 0 \quad \text { for } q>0 .  \tag{25}\\
& C_{q}^{(\alpha, \beta)} \text { is a continuous function for } q>0 .  \tag{26}\\
& \lim _{Q \uparrow \infty} C_{q}^{(\alpha, \beta)}=+\infty .  \tag{27}\\
& \lim _{Q \not 0} C_{q}^{(\alpha, \beta)}=+\infty \text {. }  \tag{28}\\
& C_{q}^{(\alpha, \beta)} \text { has an absolute minimum for } q>0 \text { at the point } q_{0}  \tag{29}\\
& \quad \text { which satisfies the equation }
\end{align*}
$$

$$
\frac{1}{\Gamma(\beta+1)} \int_{q_{0}}^{\infty}{ }_{\gamma^{\beta} e^{-v} d v=\frac{1}{2} . . . ~}^{\text {. }}
$$

Denote

$$
C^{(\alpha, \beta)}=\min _{0<q<\infty} C_{q}^{(\alpha, \beta)}
$$

Denote by $z^{\prime}$ a limit point of a sequence $\left\{s_{n}\right\}$. We denote by $z^{\prime \prime}$ a limit point, as $x \rightarrow 1$, of the Abel transform $t(x)$ of the sequence $\left\{s_{n}\right\}$. Then we obtain from Theorem 2 and Theorem 4 the following result concerning limit points $z^{\prime}$ and $z^{\prime \prime}$.

Theorem 5. Suppose the two real numbers $\alpha$ and $\beta$ satisfy $-1<\alpha$ $\leqq \beta \leqq \alpha+1$. Then for any sequence $\left\{s_{n}\right\}$ with a bounded sequence $\left\{a_{n}^{(\beta)}\right\}$ we have:
(i) To each $z^{\prime}$ corresponds at least one $z^{\prime \prime}$ such that

$$
\begin{equation*}
\left|z^{\prime}-z^{\prime \prime}\right| \leqq C^{(\alpha, \beta)} \cdot \lim _{n \rightarrow \infty} \sup _{n \rightarrow \infty}^{(\beta)} \mid \tag{30}
\end{equation*}
$$

(ii) To each $z^{\prime \prime}$ corresponds at least one $z^{\prime}$ such that

$$
\begin{equation*}
\left|z^{\prime \prime}-z^{\prime}\right| \leqq C^{(\alpha, \beta)} \cdot \limsup _{n \rightarrow \infty}\left|a_{n}^{(\beta)}\right| \tag{31}
\end{equation*}
$$

We do not know if the constant $C^{(\alpha, \beta)}$ in (30) and (31) is the best (the smallest) satisfying these inequalities.

## Bibliography

1. R. P. Agnew, Abel transforms and partial sums of Tauberian series, Ann. of Math. (2) 50 (1949), 110-117.
2. V. Garten, Über Tauber'she Konstanten bei Cesaro'schen Mittlebildung, Comment. Math. Helv. 25 (1951), 311-335.
3. P. Hartman, Tauber's theorem and absolute constants, Amer. J. Math. 69 (1947), 599-606.
4. A. Jakimovski (Amir), Some relations between the methods of summability of Abel, Borel, Cesaro, Hölder and Hausdorff, J. Analyse Math. 3 (1953/54), 346-381.
5. -, Tauberian constants for the $[J, f(x)]$ transformations, Pacific J. Math. (to appear).

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## WHITTAKER'S CONSTANT FOR LACUNARY ENTIRE FUNCTIONS

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1. Introduction. Let

$$
\begin{equation*}
f(z)=\sum_{\nu=0}^{\infty} \frac{b_{\nu}}{\nu!} z^{\nu} \tag{1}
\end{equation*}
$$

be an entire function of exponential type $\tau<\infty$. We are concerned here with two problems which are closely related to the determination of Whittaker's constant, that is to say, with theorems to the effect that if $f(z)$ and each of its derivatives have some zeros in the unit circle then $\tau$ cannot be too small.

Definition 1. The constant $W_{p}$ is the largest number $W$ for which the following assertion is true: "Let the coefficients $b_{\nu}$ of $f(z)$ vanish except for values of $\nu$ in the arithmetic progression $q, q+p, q+2 p, \cdots$. If $f(z), f^{\prime}(z), \cdots$ each have a zero in $|z|<1$, and if $\tau<W$, then $f(z) \equiv 0$."

One sees, by considering $f^{(q)}(z)$ that $W_{p}$ is independent of $q . W_{1}$ is Whittaker's constant, whose value is unknown [1]. The case $p=2$ has also been investigated $[2 ; 3]$.

Definition 2. The constant $\omega_{p}$ is the largest number $\omega$ for which the following assertion is true: "Let $f(z), f^{\prime}(z), \cdots$ each have at least $p$ zeros in $|z|<1$. If $\tau<\omega$, then $f(z) \equiv 0$."

Again $\omega_{1}=W_{1}$ is Whittaker's constant. Erdös-Rényi [6, equation (15)] have shown that

$$
\begin{equation*}
\omega_{p} \geqq p / e \quad(p=1,2, \cdots) . \tag{2}
\end{equation*}
$$

We shall give a somewhat better bound.


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