A THEOREM ON PRODUCTIVE FUNCTIONS

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It has been known for some time [1, p. 47, footnote][1] that an arbitrary productive set of natural numbers has $\aleph_0$ mutually disjoint productive centers; however, so far as we know, no proof of this fact has ever been published. It may therefore be worthwhile to exhibit a very simple proof of a somewhat more strongly formulated result.

THEOREM. *Let α be a productive set (resp., a set contraproductive under a total function); then, α admits $\aleph_0$ productive (contraproductive) functions having mutually disjoint ranges.*

PROOF. Myhill has proved that any productive set is productive under a 1-1 total function. A corresponding result is easily established for any contraproductive set having at least one total contraproductive function. Letting $h(x, y)$ be the recursive function of [2, Theorem 2.4, p. 69], consider the sequence $s_0 = h(0, 0)$, $s_1 = h(1, s_0)$, $s_2 = h(0, s_1)$, $s_3 = h(1, s_2)$, $s_4 = h(2, s_3)$, and so on. This sequence gives $\aleph_0$ mutually disjoint effective enumerations (e.g., $s_0, s_1, s_4, \cdots$) of indices of all the r.e. sets. Let $\psi_i(x)$, $0 \leqq i < \omega$, be recursive functions providing these disjoint enumerations. Then, if $p(x)$ is any 1-1 productive (contraproductive) function for α, it is straightforward to verify that (i) each function $p \circ \psi_i$ is also a productive (contraproductive) function for α, and (ii) the $p \circ \psi_i$'s have mutually disjoint ranges.

Added in proof. Since this note was submitted, the author has observed the following result: any contraproductive set admitting a *total* contraproductive function is actually productive. Thus, in particular, the hypothesis of the theorem does not cover the case of the complement of a simple or mesoic r.e. set; nevertheless, the conclusion of the theorem (for contraproductivity) does hold for such sets.

REFERENCES

1. J. C. E. Dekker and J. Myhill, *Some theorems on classes of recursively enumerable sets*, Trans. Amer. Math. Soc. 89 (1958), 25–59.

2. R. M. Smullyan, *Theory of formal systems*, Princeton Univ. Press, Princeton, N. J., 1961.

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[1] We thank Professor Dekker for drawing our attention to this reference.