POSITIVE DEFINITE MEASURES

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In this paper we prove two theorems relating positive definite measures to induced representations. The first shows how the injection of a positive definite measure on a topological group H into a containing locally compact group G in which H is closed gives rise to induced representations. The second is another version of Mackey's imprimitivity theorem, along the lines of Loomis' proof [5]. We feel this is justified on several grounds. Firstly, our proof is simpler than Loomis'. We make no use of the Radon-Nikodym theorem nor of quasi-invariant measures. Secondly, we do not assume in advance that our system of imprimitivity is based on the *reduced* algebra of Borel sets in G/H. Instead, this fact is seen as a consequence of the theorem. Finally, the statement and proof of Theorem 2 in [5] are in need of minor repairs. Using Loomis' notation, the induced representation space of V is spanned, not by the set of functions $\{f_u: u\in H\}$, but rather by the set $\{[E]f_u: u\in H,\ E$ a Borel subset of $G/K\}$. Formula (8) must then be replaced by formula (11) in the statement of the theorem. The algebra $C_0(S\times G)$ used in the present paper may be looked upon as a device for accomplishing these changes.

All nonobvious definitions, notations, and conventions are those of [1].

1. Let $f,\ g\in C_0(G)$. Define $f\circ g$ and f^* by

$$(f\circ g)(x) = \int f(y)g(xy^{-1})dy$$

and

$$f^*(x) = [f(x^{-1})]^-\delta_G(x)^{-1}.$$

$C_0(G)$, equipped with $\circ$, $*$, and the usual inductive limit topology, is a topological $*$-algebra. This is a group algebra with multiplication defined in a way differing slightly from the usual one. If $x\in G$, we define $(R(x)f)(y)=f(yx)$. The map $(x,f)\rightarrow R(x)f$ is continuous.

A measure μ on G such that $\mu(f^*\circ f)\geqq 0$ for all $f\in C_0(G)$ is called positive definite. Given such a μ, one defines a pseudo-Hilbert inner product on $C_0(G)$ by setting $(f,\ g)_\mu=\mu(g^*\circ f)$. One then completes

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$C_0(G)$ to get a Hilbert space $\mathcal{H}_\mu$ and, for each $x \in G$, extends $R(x)$ to a unitary operator $R_\mu(x)$ on $\mathcal{H}_\mu$. R_μ is then a unitary representation of G on $\mathcal{H}_\mu$.

THEOREM 1. *Let H be a closed subgroup of the locally compact group G. Let μ be a positive definite measure on H. Let ν be the measure on G obtained by injecting $\delta_H^{-1/2}\delta_G^{1/2}\mu$. Then ν is positive definite. Moreover, R_ν is unitarily equivalent to U^{R_μ} via the closure V of the map $f \to \tilde{f}$ of $C_0(G)$, where, for $x \in G$, $\tilde{f}(x)$ is the vector in $\mathcal{H}_\mu$ defined by $\tilde{f}(x)(\xi) = \delta_H(\xi)^{-1/2}\delta_G(\xi)^{1/2}f(\xi x)$.*

PROOF. Let $f \in C_0(G)$ and choose $h \geqq 0$ in $C_0(G)$ such that $\int_H h(\xi x)d\xi = 1$ for all $x \in G$ such that $f(x) \neq 0$. Then

$$\begin{aligned}
\int & (f^* \circ f)(\xi)\delta_H(\xi)^{-1/2}\delta_G(\xi)^{1/2}d\mu(\xi) \\
&= \int\int [f(x)]^- f(\xi x)\delta_H(\xi)^{-1/2}\delta_G(\xi)^{1/2}dx d\mu(\xi) \\
&= \int\int\int h(\eta x)[f(x)]^- f(\xi x)\delta_H(\xi)^{-1/2}\delta_G(\xi)^{1/2}d\eta dx d\mu(\xi) \\
&= \int\int\int h(x)[f(\eta^{-1}x)]^- f(\xi\eta^{-1}x)\delta_H(\xi)^{-1/2}\delta_G(\xi)^{1/2}\delta_G(\eta)^{-1}d\eta dx d\mu(\xi) \\
&= \int\int\int h(x)[f(\eta x)]^- f(\xi\eta x)\delta_H(\xi)^{-1/2}\delta_G(\xi)^{1/2}\delta_H(\eta)^{-1}\delta_G(\eta)d\eta dx d\mu(\xi) \\
&= \int\int\int h(x)[\tilde{f}(x)(\eta)]^- \tilde{f}(x)(\xi\eta)d\eta d\mu(\xi)dx \\
&= \int h(x)(\tilde{f}(x), \tilde{f}(x))_\mu dx \geqq 0.
\end{aligned}$$

Thus ν is positive definite.

It is trivial to verify that $\tilde{f} \in \mathfrak{F}_0$ (see [1, §2] for the definition). Therefore $\tilde{f}$ is in the Hilbert space $\mathcal{H}$ of U^{R_μ}. Moreover, the above equations, together with the definition of the norm in $\mathcal{H}$, show that $\|f\|_\nu = \|\tilde{f}\|$. Hence the isometry V is well defined. Since V clearly sets up an equivalence between R_ν and a subrepresentation of U^{R_μ}, we only have left to show that V is onto.

Let $g \in C_0(G)$, $u \in C_0(H)$. Regarding u as a member of $\mathcal{H}_\mu$, we may form

$$\epsilon(g, u)(x) = \int_H \delta_H(\xi)^{-1/2}\delta_G(\xi)^{1/2}g(\xi x)R_\mu(\xi)^{-1}u d\xi$$

as in [1]. Since this integral converges in $C_0(H)$, we obtain

$$\epsilon(g, u)(x)(\eta) = \int_H \delta_H(\xi)^{-1/2}\delta_G(\xi)^{1/2}g(\xi x)u(\eta\xi^{-1})d\xi$$

$$= \int_H \delta_H(\xi\eta)^{-1/2}\delta_G(\xi\eta)^{1/2}g(\xi\eta x)u(\xi^{-1})d\xi.$$

It is now easy to see that if we set $f(x) = \int_H \delta_H(\xi)^{-1/2}\delta_G(\xi)^{1/2}g(\xi x)u(\xi^{-1})d\xi$, then $f \in C_0(G)$ and $\tilde{f} = \epsilon(g, u)$. Thus $[\tilde{f}: f \in C_0(G)]$ is dense in $\mathfrak{H}$ by [1, Lemma 2b], and V is onto.

2. Let (S, G) be a locally compact transformation group (with G acting on the right). By a *unitary representation* of (S, G) on the Hilbert space $\mathfrak{H}$ we shall mean a *-representation E of $C_0(S)$ (under the pointwise operations) in $\mathfrak{L}(\mathfrak{H}, \mathfrak{H})$ together with a unitary representation U of G on $\mathfrak{H}$ such that:

(1) $E(C_0(S))\mathfrak{H}$ is dense in $\mathfrak{H}$;

(2) $U(x)E(f)U(x^{-1}) = E(R(x)f)$, $x \in G$, where $(R(x)f)(p) = f(px)$.

Note that from the *-representation property of E it follows that E is continuous from $C_0(S)$ in the $\|\cdot\|_\infty$ norm to $\mathfrak{L}(\mathfrak{H}, \mathfrak{H})$ in the uniform norm.

As an example, let G be a locally compact group and H a closed subgroup. Let $S = G/H$ (right cosets) and let G operate on S in the usual way. Let π be the canonical projection of G onto S. Let L be a unitary representation of H. Form the induced representation U^L of G, operating on the Hilbert space of functions $\mathfrak{H}$. For $f \in C_0(S)$, define $E^L(f)$ on $\mathfrak{H}$ by setting $(E^L(f)g)(x) = f(\pi(x))g(x)$. It is easily verified that this definition makes sense and that (E^L, U^L) is a unitary representation of (S, G) on $\mathfrak{H}$. It is called the unitary representation of (S, G) *induced* by L.

Returning now to a general transformation group, let $f, g \in C_0(S \times G)$. Define $f \circ g$ and f^* by $(f \circ g)(p, x) = \int f(p, y)g(py^{-1}, xy^{-1})dy$ and $f^*(p, x) = [f(px^{-1}, x^{-1})]^{-}\delta_G(x)^{-1}$. It is easily verified that $\circ$ and $*$ turn $C_0(S \times G)$ into a topological *-algebra (with respect to the usual inductive limit topology on $C_0(S \times G)$). Moreover, if $x \in G$ we define $(R(x)f)(p, y) = f(px, yx)$, and if $h \in C_0(S)$ we define $(P(h)f)(p, x) = h(p)f(p, x)$. It is easy to see that $(x, f) \to R(x)f$ and $(h, f) \to P(h)f$ are continuous maps from $G \times C_0(S \times G)$ into $C_0(S \times G)$ and $C_0(S) \times C_0(S \times G)$ into $C_0(S \times G)$ respectively (even when $C_0(S)$ is given the sup topology). The algebra $C_0(S \times G)$ is due to Dixmier [3] and has been studied extensively by Glimm [4].

Let (E, U) be a unitary representation of (S, G) on $\mathcal{H}$. For $f \in C_0(S \times G)$ define $\Phi(f)$ by $\Phi(f) = \int E(f(\cdot, x))U(x^{-1})dx$. It is easily verified that Φ is a continuous *-homomorphism from $C_0(S\times G)$ into $\mathcal{L}(\mathcal{H}, \mathcal{H})$ such that $\Phi(C_0(S\times G))\mathcal{H}$ is dense in $\mathcal{H}$ and such that $\Phi(R(x)f) = U(x)\Phi(f)$ and $\Phi(P(h)f) = E(h)\Phi(f)$ for all $x \in G$ and $h \in C_0(S)$. Let $v \in \mathcal{H}$ and define Λ by $\Lambda(f) = (\Phi(f)v, v)$. Λ is a Radon measure on $S\times G$ such that $\Lambda(f^* \circ f) \geqq 0$ for all $f \in C_0(S\times G)$. Any measure on $S\times G$ satisfying this positivity condition will be called *positive definite.*

Let Λ be a positive definite measure on $S\times G$. Exactly as in the case of positive definite measures on groups, we may define a pseudo-Hilbert inner production on $C_0(S\times G)$ by setting $(f, g)_\Lambda = \Lambda(g^* \circ f)$. We may then complete $C_0(S\times G)$ to get a Hilbert space $\mathcal{K}_\Lambda$. For $x \in G$, $R(x)$ extends to a unitary operator $R_\Lambda(x)$ on $\mathcal{K}_\Lambda$; for $h \in C_0(S)$, $P(h)$ extends to a bounded operator $P_\Lambda(h)$ on $\mathcal{K}_\Lambda$. (P_Λ, R_Λ) is a unitary representation of (S, G) on $\mathcal{K}_\Lambda$. If, moreover, Λ arises from a unitary representation (E, U) of (S, G) on $\mathcal{H}$ and a vector $v \in \mathcal{H}$, as above, and if $\mathcal{H}_1$ is the smallest (E, U)-invariant subspace of $\mathcal{H}$ containing v, then (P_Λ, R_Λ) is unitarily equivalent to the restriction $(E, U)|\mathcal{H}_1$ of (E, U) to $\mathcal{H}_1$ via the closure of the isometry $f \to \Phi(f)v$.

Suppose now that H is a closed subgroup of G and that $S = G/H$. For $h \in C_0(G)$, set

$$(\tau h)(\pi(x)) = \int_H h(\xi x)d\xi.$$

For $k \in C_0(G\times G\times G)$, set $(\sigma k)(\pi(x), y, z) = \int_H k(\xi x, y, z)d\xi$ and $(\theta k)(x, y, z) = k(xz, yx^{-1}, x^{-1})$. Then τ and σ are open homomorphisms of $C_0(G)$ and $C_0(G\times G\times G)$ onto $C_0(S)$ and $C_0(S\times G\times G)$ respectively, and θ is a topological automorphism of $C_0(G\times G\times G)$.

LEMMA. *Let Λ be a measure on $S\times G$. Define the measure* M *on $G\times G\times G$ by setting*

$$\iiint k(x, y, z)d\mathrm{M}(x, y, z) = \iiint (\sigma\theta k)(p, y, z)d\Lambda(p, y)dz$$

for all $k \in C_0(G\times G\times G)$. Then there is a measure μ on $G\times G$ such that $d\mathrm{M}(x, y, z) = dxd\mu(y, z)$. Moreover, for $\xi \in H$, $d\mu(y\xi, z\xi) = \delta_G(\xi)\delta_H(\xi)^{-1}d\mu(y, z)$.

PROOF. That M factors as above follows from the fact that M is invariant under right translation in its first variable (cf. the argument in [2, bottom of p. 127]). Now let $k \in C_0(G\times G\times G)$ and $\xi \in H$. Set $k^\xi(x, y, z) = k(x, y\xi^{-1}, z\xi^{-1})$ and $k_\xi(x, y, z) = k(\xi^{-1}x, y, z)$. Then

$$\begin{aligned}(\sigma\theta k^{\xi})(\pi(x), y,z) &= \int_H k(\eta xz, yx^{-1}\eta^{-1}\xi^{-1}, x^{-1}\eta^{-1}\xi^{-1})d\eta \\ &= \delta_H(\xi)^{-1}\int_H k(\xi^{-1}\eta x, yx^{-1}\eta^{-1}, x^{-1}\eta^{-1})d\eta \\ &= \delta_H(\xi)^{-1}(\sigma\theta k_{\xi})(\pi(x), y, z).\end{aligned}$$

From this we obtain

$$\begin{aligned}\iiint k(x, y, z)dxd\mu(y\xi, z\xi) &= \iiint k^{\xi}(x, y, z)dxd\mu(y, z) \\ &= \delta_H(\xi)^{-1}\iiint k_{\xi}(x, y, z)dxd\mu(y, z) \\ &= \delta_H(\xi)^{-1}\delta_G(\xi)\iiint k(x, y, z)dxd\mu(y, z)\end{aligned}$$

and our lemma is proved.

THEOREM 2. *Let Λ be a positive definite measure on $S\times G$ and define μ as in the lemma. For $\phi, \psi\in C_0(G)$, set $(\phi, \psi)_\mu = \int\phi(y)[\psi(z)]^- d\mu(y, z)$. If $\xi\in H$, set $(L(\xi)\phi)(y) = \delta_G(\xi)^{1/2}\delta_H(\xi)^{-1/2}\phi(y\xi)$. Then $(\cdot\,,\cdot)_\mu$ is a pseudo-Hilbert inner production on $C_0(G)$. Complete $C_0(G)$ to get the Hilbert space $\mathcal{V}_\mu$. Then L extends to a unitary representation L_μ of H on $\mathcal{V}_\mu$. Finally (P_Λ, R_Λ) is unitarily equivalent to (E^{L_μ}, U^{L_μ}) via the closure W of map $f\rightarrow\hat{f}$ of $C_0(S\times G)$, where, for $x\in G$, $\hat{f}(x)$ is the vector in $\mathcal{V}_\mu$ defined by $\hat{f}(x)(y) = f(\pi(x), yx)$.*

PROOF. Let $f\in C_0(S\times G)$ and $h\in C_0(G)$. Set $k(x, y, z) = h(x)[f(\pi(x), zx)]^- f(\pi(x), yx)$. Then

$$\begin{aligned}(\sigma\theta k)(\pi(x), y, z) &= \int_H h(\xi xz)[f(\pi(xz), z)]^- f(\pi(xz), yz)d\xi \\ &= (\tau h)(\pi(xz))[f(\pi(xz), z)]^- f(\pi(xz), yz).\end{aligned}$$

Hence

$$\begin{aligned}\int h(x)(\hat{f}(x), \hat{f}(x))_\mu dx &= \iiint (\tau h)(pz)[f(pz, z)]^- f(pz, yz)dzd\Lambda(p, y) \\ &= \iiint f^*(p, z)(\tau h)(pz^{-1})f(pz^{-1}, yz^{-1})dzd\Lambda(p, y) \\ &= \Lambda(f^*\circ P(\tau h)f) = (P_\Lambda(\tau h)f, f)_\Lambda.\end{aligned}$$

Now $h \geqq 0$ implies that $\tau h \geqq 0$, so that $P_\Lambda(\tau h)$ is a positive operator. Moreover, $x \to \hat{f}(x)[\hat{f}(x)]^-$ is continuous from x to $C_0(G)$ so that $x \to (\hat{f}(x), \hat{f}(x))_\mu$ is a continuous function. We conclude that $(\hat{f}(x), \hat{f}(x))_\mu \geqq 0$ for all $x \in G$. Since $f \to \hat{f}(e)$ maps $C_0(S \times G)$ *onto* $C_0(G)$, our first assertion is proved. That $L(\xi)$ is unitary for all $\xi \in H$ follows from the lemma, and that L_μ is a unitary representation of H is then clear.

Now it is easy to see that $\hat{f} \in \mathfrak{F}_0$. Choose $h \in C_0(G)$ so that $\tau h = 1$ on $[p \in S: f(p, y) \neq 0$ for some $y \in G]$. Then $P_\Lambda(\tau h)f = f$ and we obtain $\|f\|_\Lambda = \|\hat{f}\|$. Once again we are reduced to showing that W is onto. This is done exactly as in Theorem 1. Let g, $u \in C_0(G)$. Regarding u as a member of $\mathfrak{V}_\mu$, form $\epsilon(g, u)$. We obtain $\epsilon(g, u)(x)(y) = \int_H g(\xi x)u(y\xi^{-1})d\xi$. Set $f(\pi(x), y) = \int_H g(\xi x)u(yx^{-1}\xi^{-1})d\xi$. If the supports of g and u are K_1 and K_2 respectively, the support of f is contained in $\pi(K_1) \times (K_2K_1)$, compact. Hence $f \in C_0(S \times G)$. It is easy to see that $\hat{f} = \epsilon(g, u)$, and our proof finishes as before.

COROLLARY. *Every unitary representation of* $(G/H, G)$ *is induced.*

PROOF. If the representation space is jointly cyclic under E and U' the corollary follows from the theorem together with the remarks two paragraphs before the lemma. The general case follows from the fact that induction commutes with direct summation.

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