INTERVAL CLANS WITH NONDEGENERATE KERNEL[1]

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Introduction. The object of this paper is to characterize the clans (compact connected Hausdorff topological semigroups with an identity element) which are homeomorphic to a unit interval and which have a nondegenerate kernel (minimal two-sided ideal). The corresponding case when the kernel is degenerate has been characterized in a paper by H. Cohen and L. I. Wade [**2**] together with an earlier paper by Mostert and Shields [**5**].

In a topological semigroup T, $K(T)$ or K denotes the kernel of T. The symbol u is reserved to denote an identity element. The term "standard thread" will mean a clan with zero which is homeomorphic to a unit interval and whose endpoints are its zero and identity element. In a standard thread T with identity element u and zero 0, for $a, b \in T$, $[a, b]$ will denote the interval from a to b, (or b to a) inclusive and $a \leqq b$ will mean $a \in [0, b]$, with $a < b$ in case $a \neq b$. A relation R on a topological semigroup x is called a "closed right congruence" if (i) R is an equivalence relation, (ii) $a, b, c \in X$, aRb implies $acRbc$, (iii) aRx_n for $n = 1, 2, \cdots$ and $x_n \to x$ implies aRx (closed). We denote by R_a the set $\{x \mid aRx\}$. The analog of Theorem I where R satisfies (2)′ ($a, b, c \in X$, aRb implies $caRcb$ (left congruence) instead of (2) is also true.

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THEOREM I. *Let T be a standard thread and R_a closed right congruence on T. Then for $a \in T$ either* (1) $R_a = a$ *or* (2) R_a *is an interval* $[e, b]$ *where e is idempotent and* $[e, b]$ *is a subsemigroup of T with zero element e.*

PROOF. Suppose there exists $a' \in T$ such that $a'Ra$ and $a' \neq a$. Let $e = \inf\{x : xRa\}$ and $b = \sup\{x : xRa\}$. Since R_a is closed, eRa and bRa. Now $e < b$ which implies [**3**] that $e = br$ for some $r \geqq e$. Therefore, bRe implies $brRer$ implies $eRer$ implies $eRer^n$ for $n = 1, 2, \cdots$. From [**3**] we know $r^n \to j = j^2 \leqq r$ and hence $eRej$. We will show $j \leqq e$. If $j \geqq b$, then $bj = b$ and hence $b = bj = b(jr) = (bj)r = br = e$. Therefore $j < b$

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and $j=bj\leqq br=e$, as was to be shown. Now $j\leqq e$ implies $ej=j$ and since $eRej$, eRj which implies by the minimality of e, that $e=j$ and hence $e^2=e$. Now for $c\in[e, b]$, $c=bp$ for some $p\geqq c$ and since eRb, $epRbp$ and we have eRc. This shows that $R_a=[e, b]$ which is indeed a semigroup with zero element e.

Notice that if X and Y are topological semigroups and f maps X into Y continuously and either $f(xy)=f(x)f(y)$ or $f(xy)=f(x)\cdot y$, (in case Y is contained in X), then the relation R induced by letting $R_a=f^{-1}f(a)$ is a closed right congruence on X.

COROLLARY II. *Let X be a topological semigroup and let A be a standard thread contained in X. Then for any element $c\in X$, cA is a continuous monotone image of A.*

PROOF. Define the relation R on A by $R_a=\{x \mid x\in A,\ cx=ca\}$. Then R is a closed right congruence and hence multiplication by c is monotone.

Clans on an interval. Let S be a clan which is homeomorphic to a unit interval and which has a nondegenerate kernel, K. By a result of A. D. Wallace [7], the identity element u of S is one of the endpoints of S. Note that K is a closed interval of S and let (i) $A=$ the closure of the component of $S-K$ which contains u, (ii) $B=S-(K^0\cup A)$, (iii) $z=A\cap K$, (iv) $z'=B\cap K$, and (v) $d=$ the nonidentity endpoint of S. A result of Faucett [4] is that A is an abelian subclan of S with zero z and that K consists of either all left zeroes or all right zeroes of S. Let us assume that K is all right zeroes.

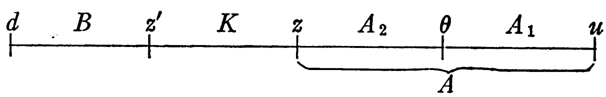

FIGURE I

LEMMA III. $AB=Ad=B$.

PROOF. Since A is a standard thread contained in S, by the analog of Corollary II, Ad is a continuous monotone image of A and $Ad=[zd, ud]=[zd, d]$ which contains B since zd is in K. In particular, there exists an element a in A such that $z'=ad$. Now since z is the zero for A, $z=za$ and we have $zd=(za)d=z(ad)=zz'=z'$. Therefore $Ad=[d, z']=B$. Also $AB=A(Ad)=A^2d=Ad=B$ as was to be shown.

Notice that from Corollary II we have $dA=[dz, du]=[z, d]$ which contains z'. Let $\theta=\inf\{a: a\in A,\ da=z'\}$. Denote $[\theta, u]$ by A_1 and $[z, \theta]$ by A_2. (Note that $\theta\neq z$, else $z'=d\theta=z$.)

LEMMA IV. $BA_1=dA_1=B$; $BA_2=dA_2=K$; *and* $\theta^2=\theta$.

PROOF. In the same manner as the proof of Lemma III we have $dA_1=[z', d]=B$ and $dA_2=[z', z]=K$. Since $A_2^2\subset A_2$[3], $\theta^2\in A_2$. Therefore $d\theta^2\in dA_2=K$. But $d\theta^2=z'\theta\in BA_1=AdA_1=AB=B$. So $d\theta^2\in K\cap B=z'$ and by the minimality of θ, $\theta^2=\theta$. Now we employ a result of Mostert and Shields [5] that A_1 and A_2 are subclans of A, that θ is a zero for A_1 and an identity for A_2, and that $a\in A_1$, $a'\in A_2$ implies $aa'=a'a=a'$. Therefore $A_1A_2=A_2$ and $BA_2=dA_1A_2=K$ and the lemma is proved.

LEMMA V. *B is an abelian subsemigroup of S.*

PROOF. First we show d^2 is in B. Clearly d^2 is not in A, else $z'=dz'=d(d\theta)=d^2\theta\in A$. So suppose d^2 is in K. Then $dd^2=d^2$ and by Lemma IV $d^2=da$ for some a in A_2. Therefore $d^2u=d^2=dd^2=d(da)=d^2a$, and since left multiplication by d^2 induces a closed right congruence on $[z, u]$, $d^2[u, a]=d^2$. Now $a\in A_2$ so that θ is in $[u, a]$. Hence $d^2\theta=d^2$. But $d^2\theta=d(d\theta)=dz'=z'$ and d^2 is in B. Using Lemmas III and IV, we have $B^2=(dA_1)(dA_1)\subset dBA_1=d(dA_1)\subset BA_1=B$, i.e., $B^2\subset B$. Now we show B is abelian. Using again Lemmas III and IV, $a\in A_1$ implies $da\in B$ and $da=a'd$ for some $a'\in A$. Since $d^2\in B$, $d^2=a''d$ for some $a''\in A$. Using the commutativity of A, we have $dad=(a'd)d=a'a''d=a''a'd=a''da=d^2a$. Let b, b' be elements of B. Then for some $a_1, a_2\in A_1$, $b=da_1$, and $b'=da_2$. So $bb'=da_1da_2=d^2a_1a_2=d^2a_2a_1=da_2da_1=b'b$ and B is abelian.

LEMMA VI. *$KB=z'$.*

PROOF. Since $KB=dA_2dA_1=d(A_2d)A_1\subset dBA_1=dB\subset B$, $KB\subset K\cap B=z'$.

In what follows S/K denotes the Rees quotient [6] of S modulo K and F denotes the natural map of S onto S/K. Since $d^2\in B$, $F(d^2)\in F(B)$ and we have

THEOREM VII. *Let S be an interval clan with a nondegenerate kernel K. Let u be the identity and d the nonidentity endpoint of S/K; denote $F(K)$ by 0. Then* (i) *there exists an element $\theta=\theta^2\in[0, u]-\{0\}$ such that $d\theta=0$ and* (ii) *$d^2\in[d, 0]$. Further, the function h: $F(A_2)\to K$ defined by $h(x)=d\cdot F^{-1}(x)$ for $x\neq 0$ and $h(0)=z$ is continuous and induces a closed right congruence on $F(A_2)$.*

Let S be a clan on an interval $[d, u]$ where u is the identity element. Suppose S has a zero 0, that (i) and (ii) of Theorem VII are satisfied, that $d\neq 0$ and that $u\neq\theta$. Consider the real interval $[1, 5]$ and define

1. f: $[d, 0]\to[1, 2]$ so that $f(d)=1$, $f(0)=2$ and f is a homeomorphism,

2. g: $[0, u] \to [3, 5]$ so that $g(0) = 3$, $g(\theta) = 4$, $g(u) = 5$ and g is a homeomorphism,

3. h: $[0, \theta] \to [2, 3]$ so that $h(0) = 3$, $h(\theta) = 2$, h is continuous and h induces a closed right congruence R on $[0, \theta]$, $(R_x = h^{-1}h(x))$.

Further, define for all x, y and z on which the functions are defined,

4. $c \cdot h(x) = h(x)$, all $c \in [1, 5]$,
5. $h(x) \cdot g(y) = h(xy)$,
6. $h(x) \cdot f(y) = 2$,
7. $g(x) \cdot g(y) = g(xy)$,
8. $f(x) \cdot f(y) = f(xy)$,
9. $g(x) \cdot f(y) = f(xy)$,
10. $f(x) \cdot g(y) = \{h(y)$ for $y \in [0, \theta]$, $f(xy)$ for $y \in [\theta, u]\}$.

We now show that definition 5 is well defined. The others are clear. Suppose $h(a) = h(b)$ and $a < b$. Then from condition 3 and Theorem I we have $a, b \in [e, r] = h^{-1}h(x)$ and $e^2 = e$ and e is a zero for $[e, r]$, a semigroup. Then for $c \in [0, u]$, $h(a) \cdot g(c) = h(ac)$ and $h(b) \cdot g(c) = h(bc)$. If $c \leqq e$, then $ac = bc = c$ [5] so that $h(ac) = h(bc)$. If $c > e$, then $e \leqq ac \leqq a$ and $e \leqq bc \leqq b$ and $h(ac) = h(bc) = h(a)$, and h is well defined.

It can be easily verified that the interval $[1, 5]$ together with definitions 4 through 10 is a clan S' with kernel $[2, 3]$ and that S'/K is topologically isomorphic to S.

Now suppose S in the previous construction were the Rees quotient of an interval clan $T = [d, z', z, u]$ with nondegenerate kernel $K = [z', z]$. Then outside K, T is reproduced by our construction and if h in 3 is chosen to be the h of Theorem VII, the resulting clan is topologically isomorphic to T.

If $d = 0$, omit definitions 1, 6, 8, 9, and 10; if also $u = \theta$, omit the equation $g(u) = 5$ from definition 2; change definition 4 appropriately. The conclusion is completely analogous for $S = [2, 4]$ if $u = \theta$ or $S = [2, 5]$ if $u \neq \theta$.

From the preceding two paragraphs we conclude:

THEOREM VIII. *An interval clan with a nondegenerate kernel is characterized by a pair* (S, h) *where*

1. *S is an interval clan with zero* 0, *say* $S = [d, 0, u]$ *where d may equal* 0,
2. $d^2 \in [d, 0]$,
3. *there exists* $\theta = \theta^2 \in (0, u]$ *such that* $d\theta = 0$,
4. *h maps* $[0, \theta]$ *onto* $[1, 2]$ *continuously,*
5. *h induces a closed right* (*left in case K is all left zeroes*) *congruence on* $[0, \theta]$.

The pair (S, h) characterizes a particular interval clan T with a nondegenerate kernel K in the following sense; $T/K = S$ and the function h of Theorem VII satisfy the conditions of Theorem VIII, and a clan T' gives rise to the same S and h if and only if T' is topologically isomorphic to T.

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