ON ASYMMETRIC ENTIRE FUNCTIONS

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Let $p(z)$ be a polynomial of degree n having all its zeros in $|z| \leqq 1$. Then according to Walsh's generalization of Laguerre's theorem [3, Lemma 1, p. 13]

$$\frac{p'(e^{i\theta})}{p(e^{i\theta})} = \frac{n}{e^{i\theta} - w}$$

for points $e^{i\theta}$ other than zeros of $p(z)$ where $|w| \leqq 1$. Hence $|e^{i\theta} - w| \leqq 2$ and

$$\left|\frac{p'(e^{i\theta})}{p(e^{i\theta})}\right| \geqq \frac{n}{2}.$$

If $\max_{0 \leq \theta < 2\pi} |p(e^{i\theta})| = |p(e^{i\theta_0})|$, then

$$(1) \quad \max_{0 \leqq \theta < 2\pi} |p'(e^{i\theta})| \geqq |p'(e^{i\theta_0})| \geqq \frac{n}{2} |p(e^{i\theta_0})| = \frac{n}{2} \max_{0 \leqq \theta < 2\pi} |p(e^{i\theta})|.$$

We shall obtain a result for entire functions which generalizes (1). To see what to expect, note that $p(e^{iz})$ is an entire function $f(z)$ of exponential type of a special kind: if $h(\theta)$ is its indicator, we have $h(-\pi/2) = n$, but $h(\pi/2) \leqq 0$. If $p(z)$ has no zeros in $|z| > 1$, $f(z)$ has no zeros in $y < 0$.

Let us consider, then, entire functions $f(z)$ of exponential type τ with $\text{l.u.b.}_{-\infty<x<\infty} |f(x)| = 1$, $h(-\pi/2) = n$, $h(\pi/2) \leqq 0$, and $f(z) \neq 0$ for $y < 0$.

THEOREM. $\text{l.u.b.}_{-\infty<x<\infty} |f'(x)| \geqq \tau/2$.

To prove the theorem put $g(z) = f(z)e^{-i\tau z/2}$. Then $\text{l.u.b.}_{-\infty<x<\infty} |g(x)| = 1$ and $g(z)$ is of exponential type $\tau/2$; moreover the indicator h_g of g satisfies $h_g(-\pi/2) \geqq h_g(\pi/2)$. Since $g(z)$ has no zeros for $y < 0$ it belongs to the class P discussed in [1, pp. 129–131] and can be represented in the form

$$g(z) = Az^m e^{cz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left\{z \operatorname{Re}\left(\frac{1}{z_n}\right)\right\}$$

where $\operatorname{Im}(z_n) \geqq 0$ and $2 \operatorname{Im} c = h_g(-\pi/2) - h_g(\pi/2) \geqq 0$. Thus for $-\infty < x < \infty$

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$$\operatorname{Im}\left\{\frac{g'(x)}{g(x)}\right\} = \operatorname{Im} c + \sum_{n=1}^{\infty} \frac{b_n}{(x-a_n)^2+(y-b_n)^2},$$

where $z_n = a_n + ib_n$, $b_n \geqq 0$. The right hand side is non-negative. Hence

$$\operatorname{Im}\left\{\frac{f'(x)}{f(x)}\right\} = \operatorname{Im}\left\{\frac{g'(x)}{g(x)}\right\} + \frac{\tau}{2} \geqq \frac{\tau}{2}\cdot$$

Let ϵ be any number >0. There exists a number x_0 such that $-\infty < x_0 < \infty$, and $|f(x_0)| > 1-\epsilon$. So that

$$\underset{-\infty<x<\infty}{\text{l.u.b.}} |f'(x)| \geqq |f'(x_0)| = |f(x_0)| \left|\frac{f'(x_0)}{f(x_0)}\right| \geqq (1-\epsilon)\frac{\tau}{2}\cdot$$

Making $\epsilon \to 0$ we get the result.

A theorem of Boas [2, Theorem 2] states that if $f(z)$ is an entire function of exponential type τ with $|f(x)| \leqq 1$ for real x, $h(-\pi/2) = \tau$, $h(\pi/2) = 0$, and $f(z) \neq 0$ for $y > 0$, then for real x

$$|f'(x)| \leqq \frac{\tau}{2}\cdot$$

Combining this result with the conclusion of our theorem we obtain the following

Corollary. *If $f(z)$ is an entire function of exponential type τ with* $\text{l.u.b.}_{-\infty<x<\infty} |f(x)| = 1$, $h(-\pi/2) = \tau$, $h(\pi/2) = 0$, *and $f(z)$ has all its zeros on the real axis, then*

$$\underset{-\infty<x<\infty}{\text{l.u.b.}} |f'(x)| = \frac{\tau}{2}\cdot$$

References

1. R. P. Boas, Jr., *Entire functions*, Academic Press, New York, 1954.
2. ———, *Inequalities for asymmetric entire functions*, Illinois J. Math. 1 (1957), 94–97.
3. J. L. Walsh, *The location of critical points of analytic and harmonic functions*, Amer. Math. Soc. Colloq. Publ. Vol. 34, Amer. Math. Soc., Providence, R. I., 1950.

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