A GENERALIZATION OF ABSOLUTE RIESZIAN SUMMABILITY

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1. **Introduction.** Absolute Rieszian summability was defined in 1928 by N. Obreschkoff [4; 5] as follows:

DEFINITION 1. Let $k>0$, and $0\leqq\lambda_1<\lambda_2<\cdots<\lambda_n$, $\lambda_n\rightarrow\infty$ as $n\rightarrow\infty$. Let

$$C_\lambda^k(\omega) = \sum_{\lambda_n\leqq\omega} a_n\left(1-\frac{\lambda_n}{\omega}\right)^k.$$

If the integral

$$\int_a^\infty \left|\frac{d}{d\omega} C_\lambda^k(\omega)\right| d\omega < \infty, \qquad a \geqq 0,$$

then $\sum a_n$ is said to be absolutely summable by Rieszian means of order k and type λ, or summable $|R, \lambda, k|$.

The case $\lambda_n=n$ is of particular interest in this paper. Summability $|R, n, k|$ has been shown by J. M. Hyslop [3] to be equivalent to absolute Cesàro summability of order k, or summability $|C, k|$. One of the principal results shown by Obreschkoff was the consistency of the $|R, n, k|$ means; that is, he showed that summability $|R, n, k|$ implies summability $|R, n, k'|$, where $k'>k$.

In this paper we introduce a method of absolute summability based upon the (α, β) method of summability defined by Bosanquet and Linfoot [1]. Just as the Bosanquet-Linfoot method generalized Riesz's arithmetic mean (R, n, α), the method given here will generalize absolute Rieszian summability $|R, n, \alpha|$.

DEFINITION 2. A series $\sum a_n$ is said to be absolutely summable (α, β), or summable $|\alpha, \beta|$, where $\alpha>0$ or $\alpha=0$, $\beta>0$, if for each sufficiently large C,

$$\int_0^\infty \left|\frac{d}{d\omega} A_{\alpha,\beta}(\omega)\right| d\omega < \infty, \tag{1}$$

where

$$A_{\alpha,\beta}(\omega) = \sum_{n<\omega} B\left(1-\frac{n}{\omega}\right)^\alpha \log^{-\beta}\frac{C}{1-n/\omega}\, a_n, \tag{2}$$

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and $B=\log^\beta C$. Summability $|0, 0|$ is defined to be absolute convergence.

Thus $|\alpha, 0|$ summability is the same as $|R, n, \alpha|$ summability. Condition (1) is equivalent to the bounded variation of $A_{\alpha,\beta}(\omega)$ in $(0, \infty)$. (See [2, p. 605].)

In the present paper it will be proved that $|\alpha, \beta|$ summability is consistent in the following sense: $|\alpha, \beta|$ summability implies $|\alpha', \beta'|$ summability, where either $\alpha'>\alpha$ or $\alpha'=\alpha$, $\beta'>\beta$. In a future paper, the authors propose to show some applications of $|\alpha, \beta|$ summability analogous to known results for absolute Rieszian, or Cesàro, summability.

2. Lemmas.

LEMMA 1. *Let $f(x)$, $k(u)$, and $K(u)$ satisfy the following conditions:*

(i) *For some $n\geqq 0$, $V_0^T(x^{-n}f(x))<\infty$ for all $T>0$. (It will be assumed throughout that for $x=0$, the function $x^{-n}f(x)$ is replaced by $\lim_{x\to+0} x^{-n}f(x)$.)*

(ii) *$k(u)$ is absolutely continuous in $[0, 1]$.*

(iii) *$K(u)$ is positive, continuously differentiable in $[0, 1)$, Lebesgue integrable over $[0, 1]$, $\lim_{u\to 1-} K(u)=+\infty$ and $uK'(u)/K(u)$ is non-decreasing.*

Let

$$F(x) = x^{-n}\int_0^1 K(u)f(xu)du; \qquad G(x) = x^{-n}\int_0^1 k(u)K(u)f(xu)du.$$

Then $V_0^\infty G(x)\leqq\gamma V_0^\infty F(x)$, where $\gamma=\int_0^1|k'(u)|du+|k(1)|$.

PROOF. For $T>0$ let p be a partition, $0=x_0<x_1<\cdots<x_N=T$, of $[0, T]$. Corresponding to this partition let us define

$$\Delta(f_i, u) = x_i^{-n}f(ux_i) - x_{i-1}^{-n}f(ux_{i-1})$$

and

$$\Delta G_i = G(x_i) - G(x_{i-1}), \qquad i = 1, 2, \cdots, N.$$

Then

$$\sum_{(p)} |\Delta G_i| = \sum_{(p)}\left|\int_0^1 k(u)K(u)\Delta(f_i, u)du\right|. \tag{3}$$

An integration by parts of the right side of (3) leads to the inequality

$$\sum_{(p)} |\Delta G_i| \leqq C_1 + \int_0^1 |k'(u)| \sum_{(p)}\left|\int_0^u K(t)\Delta(f_i, t)dt\right| du, \tag{4}$$

where $C_1 = |k(1)|\, V_0^T F(x)$. Since $\sum_{(p)} \left|\int_0^u K(t)\Delta(f_i, t)dt\right|$ is a continuous function of u, (4) becomes, with the aid of the first mean-value theorem,

$$\sum_{(p)} |\Delta G_i| \leqq C_1 + C_2 \sum_{(p)} \left| \int_0^{u_0} K(u)\Delta(f_i, u)du \right|, \tag{5}$$

where $C_2 = \int_0^1 |k'(u)|\, du$ and $0 \leqq u_0 \leqq 1$.

If $u_0 = 0$ or 1, the right side of (5) is clearly no greater than $\gamma V_0^T F(x)$, where $\gamma = \int_0^1 |k'(u)|\, du + |k(1)|$. If $0 < u_0 < 1$, then after changing variables and integrating by parts, (5) becomes

$$\begin{aligned} \sum_{(p)} |\Delta G_i| \leqq C_1 + C_2 \Bigg\{ \sum_{(p)} \Bigg| & \left[\frac{u_0 K(uu_0)}{K(u)} \int_0^u K(t)\Delta(f_i, tu_0)dt \right]_{u=0}^{u=1} \\ & - u_0 \int_0^1 \frac{d}{du}\left(\frac{K(uu_0)}{K(u)}\right) \int_0^u K(t)\Delta(f_i, tu_0)dtdu \Bigg| \Bigg\}. \end{aligned} \tag{6}$$

But hypothesis (iii) implies that the integrated part vanishes at both limits, and that $(d/du)\{K(uu_0)/K(u)\} \leqq 0$. Again applying the first mean-value theorem, it follows from (6) that

$$\sum_{(p)} |\Delta G_i| \leqq C_1 + C_2 u_0 \sum_{(p)} \left| \int_0^{u_1} K(u)\Delta(f_i, uu_0)du \right|, \tag{7}$$

where $0 \leqq u_1 \leqq 1$.

Repetition of the steps leading from (5) to (7) gives the result,

$$\sum_{(p)} |\Delta G_i| \leqq C_1 + C_2 \Pi_m \sum_{(p)} \left| \int_0^1 K(uu_m)\Delta(f_i, u\Pi_m)du \right|, \tag{8}$$

where

$$\Pi_m = \prod_{\nu=0}^{m} u_\nu, \quad 0 \leqq u_\nu \leqq 1, \quad u_\nu \neq 0, 1 \text{ for } \nu < m, \quad m = 1, 2, \cdots.$$

From (8) we shall deduce that

$$\sum_{(p)} |\Delta G_i| \leqq \gamma V_0^T F(x). \tag{9}$$

There are two cases to consider.

Case 1. For some m, either $u_m = 0$ or 1. It is not difficult to verify then that $\sum_{(p)} |\Delta G_i| \leqq C_1$, or $\sum_{(p)} |\Delta G_i| \leqq C_1 + C_2(\Pi_m)^{n+1} V_0^{T\Pi_m} F(x)$, respectively. In either case (9) is clearly satisfied. This case for $m=0$ has been settled already.

Case 2. Suppose $u_m \neq 0$, 1 for all m. Since $\{\Pi_m\}$ is a monotone sequence, $\Pi_m \to L$ as $m \to \infty$, $0 \leqq L < 1$. If $L = 0$ then

$$\Pi_m \sum_{(p)} \left| \int_0^1 K(uu_m)\Delta(f_i, u\Pi_m)du \right| \leqq 2MN(\Pi_m)^{n+1} \int_0^1 K(u)du = o(1)$$

as $m \to \infty$, where $M = \text{l.u.b. } [x^{-n}f(x)]$ over $[0, T]$. Hence (9) holds when $L = 0$.

Finally, if $L \neq 0$, then necessarily $\lim_{m\to\infty} u_m = 1$. Since each integrand in (8) is majorized by a summable function, a well-known theorem of Lebesgue integration may be applied to (8) to give

$$\begin{aligned} \sum_{(p)} |\Delta G_i| &\leqq C_1 + C_2 L \sum_{(p)} \left| \int_0^1 K(u)\Delta(f_i, uL)du \right| \\ &\leqq C_1 + C_2 L^{n+1} V_0^{TL} F(x) \\ &\leqq \gamma V_0^T F(x). \end{aligned}$$

Thus the truth of (9) has been established for each partition p and each $T > 0$. From (9) it follows that $V_0^T G(x) \leqq \gamma V_0^T F(x)$, and from this the lemma.

LEMMA 2. *Lemma* 1 *remains valid if condition* (iii) *is replaced by*: (iii)*. $K(u)$ *is constant in* $[0, 1]$.

PROOF. An argument similar to that in the preceding lemma will show that (8) also holds under (iii)*. Then (9) is easily verified, and the conclusion follows.

3. The consistency theorem.

THEOREM. *If* $\sum a_n$ *is summable* $|\alpha, \beta|$, *then it is summable* $|\alpha', \beta'|$, *for* $\alpha' > \alpha$, *or* $\alpha' = \alpha$, $\beta' > \beta$.

PROOF.

Case 1. $\alpha = \beta = 0$. We must show that absolute convergence of the series implies $|\alpha', \beta'|$ summability, where $\alpha' > 0$ or $\alpha' = 0$, $\beta' > 0$. Let

$$\begin{aligned} \Phi_{\alpha,\beta}(u) &= Bu^\alpha \log^{-\beta} \frac{C}{u}, \qquad \text{if } u \neq 0, \\ \Phi_{\alpha,\beta}(0) &= 0, \quad \text{if } \alpha > 0 \quad \text{or} \quad \alpha = 0,\ \beta > 0. \end{aligned} \tag{10}$$

Then, what we have to show is the convergence of the integral

$$\int_0^\infty \left| \frac{1}{\omega^2} \sum_{n<\omega} \Phi'_{\alpha',\beta'}\left(1 - \frac{n}{\omega}\right) na_n \right| d\omega.$$

Noting that for $n<\omega$, $\Phi'_{\alpha',\beta'}(1-n/\omega)>0$ for sufficiently large C, we have[1]

$$\begin{aligned}\int_0^\infty &\left|\frac{1}{\omega^2}\sum_{n<\omega}\Phi'_{\alpha',\beta'}\left(1-\frac{n}{\omega}\right)na_n\right|d\omega\\ &\leqq \int_0^\infty \sum_{n<\omega}|a_n|\frac{n}{\omega^2}\Phi'_{\alpha',\beta'}\left(1-\frac{n}{\omega}\right)d\omega\\ &\leqq \sum_{n=0}^\infty |a_n|\int_n^\infty \frac{n}{\omega^2}\Phi'_{\alpha',\beta'}\left(1-\frac{n}{\omega}\right)d\omega\\ &= \sum_{n=0}^\infty |a_n|\int_0^1 \Phi'_{\alpha',\beta'}(u)du\\ &= \sum_{n=0}^\infty |a_n|.\end{aligned}$$

The result now follows, since $\sum|a_n|$ is finite.

Case 2. $\alpha>0$, or $\alpha=0$, $\beta>0$. In this case it is known [1, p. 209] that $A_{\alpha,\beta}(\omega)$ has the integral representation,

$$A_{\alpha,\beta}(\omega) = \int_0^1 \Phi'_{\alpha,\beta}(1-u)A(\omega u)du,$$

where $A(x)=\sum_{n\leqq x}a_n$. Let $h=[\alpha]$; then as in [1, p. 216] $A_{\alpha,\beta}(\omega)$ may be written in the following forms:

$$A_{\alpha,\beta}(\omega) = \omega^{-j}\int_0^1 \Phi^{(j+1)}_{\alpha,\beta}(1-u)A_j(\omega u)du, \tag{11}$$

for $j=0, 1, \cdots, h$, if $\alpha=h$, $\beta>0$ or $h<\alpha<h+1$; for $j=0, 1, \cdots, h-1$, if $\alpha=h\geqq 1$, $\beta\leqq 0$; where $A_j(x)=\int_0^x A_{j-1}(t)dt$ and $A_0(x)=A(x)$.

By choosing the appropriate form in (11), one finds that for $\alpha'>\alpha$ or $\alpha'=\alpha$, $\beta'>\beta$,

$$A_{\alpha',\beta'}(\omega) = \omega^{-(h-1)}\int_0^1 \frac{\Phi^{(h)}_{\alpha',\beta'}(1-u)}{\Phi^{(h)}_{\alpha,\beta}(1-u)}\Phi^{(h)}_{\alpha,\beta}(1-u)A_{h-1}(\omega u)du \tag{12}$$

when $\alpha=h$, $\beta\leqq 0$, and

$$A_{\alpha',\beta'}(\omega) = \omega^{-h}\int_0^1 \frac{\Phi^{(h+1)}_{\alpha',\beta'}(1-u)}{\Phi^{(h+1)}_{\alpha,\beta}(1-u)}\Phi^{(h+1)}_{\alpha,\beta}(1-u)A_h(\omega u)du \tag{13}$$

when $\alpha=h$, $\beta>0$ or $h<\alpha<h+1$.

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[1] For justification of interchange of order of summation and integration, see, e.g., Titchmarsh [6, p. 348].

A routine calculation shows that the first and second factors of the integrands (12) and (13) satisfy the requirements for $k(u)$ and $K(u)$, respectively, in Lemma 1 or 2 (whichever is applicable) for C sufficiently large. The theorem now follows immediately from these two lemmas.

References

1. L. S. Bosanquet and E. H. Linfoot, *Generalized means and the summability of Fourier series*, Quart. J. Math. Oxford Ser. (2) **2** (1931), 207–229.

2. E. W. Hobson, *The theory of functions of a real variable.* I, 3rd. ed., Harren Press, Washington, D. C., 1950.

3. J. M. Hyslop, *On the absolute summability of series by Rieszian means*, Proc. Edinburgh Math. Soc. (2) **5** (1936), 46–54.

4. N. Obreschkoff, *Sur la sommation absolue des series de Dirichlet*, C.R. Acad. Sci. Paris **186** (1928), 215–217.

5. ———, *Über die absolute Summierung der Dirichletschen Reihen*, Math. Z. **30** (1929), 375–386.

6. E. C. Titchmarsh, *The theory of functions*, 2nd ed., Oxford Univ. Press, London, 1939.

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