ON THE SPECTRAL SYNTHESIS OF BOUNDED FUNCTIONS

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0. In this note, we intend to generalize a theorem on the spectral synthesis of bounded functions due to A. Beurling [2] and to discuss an analogous problem in the case of bounded sequences. For all terms not explained here, the reader is referred to the papers of A. Beurling [2] and J. P. Kahane [4].

1. For a function $\phi(x) \in L^\infty(-\infty, \infty)$, we shall denote its spectral set by Sp.(ϕ). We shall be concerned with the space $\mathfrak{A}$ of Fourier transforms of functions in $L^1(-\infty, \infty)$. That is, $f(t) \in \mathfrak{A}$ means that there exists a function $F(x) \in L^1(-\infty, \infty)$ whose Fourier transform is $f(t)$. We introduce a norm $\|f\|_{\mathfrak{A}}$ in the space $\mathfrak{A}$ defining it by

$$\|f\|_{\mathfrak{A}} = \int_{-\infty}^{\infty} |F(x)| \, dx.$$

We say $g(t) = T(f(t))$ (or T) is a normalized contraction of f when the complex function $T(z)$ satisfies the Lipschitz condition $|T(z) - T(z')| \leqq |z - z'|$ and $\lim_{t\to\infty} g(t) = 0$. Moreover, we say an element f of $\mathfrak{A}$ is contractible in the space $\mathfrak{A}$ if every normalized contraction of f also belongs to the space $\mathfrak{A}$. And we say f is uniformly contractible in the space $\mathfrak{A}$ if f is contractible in $\mathfrak{A}$ and if $\lim_{n\to\infty} \|g_n\|_{\mathfrak{A}} = 0$ for any sequence $\{g_n(t)\}$ of normalized contractions of $f(t)$ such that $\lim_{n\to\infty} g_n(t) = 0$.

Some of Beurling's theorems [2] may be read as follows:

THEOREM 1. *Suppose that*

(i) $f(t)$ *is uniformly contractible in the space* $\mathfrak{A}$, *and*

(ii) $f(t) = \int_{-\infty}^{\infty} e^{ixt} F(x) dx$ *vanishes on* Sp.(ϕ).

Then we have

(iii) $\int_{-\infty}^{\infty} \phi(x) F(x) dx = 0$.

THEOREM 2. *Suppose that*

(iv) $w(x) \in L^1(-\infty, \infty)$ *and* $w(x)$ *is even, positive and nonincreasing in* $(0, \infty)$, *and*

(v) $|F(x)| \leqq w(|x|)$.

Then the Fourier transform f of F is uniformly contractible in the space $\mathfrak{A}$.

Note that the assumptions (ii), (iv) and (v) imply the conclusion (iii). This means that the spectral synthesis of a bounded function ϕ

Received by the editors April 4, 1962.

is possible with respect to the topology defined by the norm $\|\psi\|_{\infty,w} = \int_{-\infty}^{\infty} |\psi(x)| w(x)dx$ for $\psi \in L^\infty(-\infty, \infty)$.

The first purpose of this note is to generalize the assumption (iv). In fact, we shall prove the following:

THEOREM 3. *The assumption* (iv) *in Theorem* 2 *can be replaced by the following conditions*:

(iv)* $w(x)$ *is even, positive and satisfies*

$$\int_0^\infty x^{-3/2}\left(\int_0^x u^2 w^2(u)du\right)^{1/2} dx + \int_0^\infty x^{-1/2}\left(\int_x^\infty w^2(u)du\right)^{1/2} dx < \infty.$$

PROOF OF THEOREM 3. In a previous paper (M. Kinukawa [5]), we proved that f is contractible in $\mathfrak{A}$ under the assumptions of Theorem 3. That is, for any normalized contraction $g(t)$ of $f(t)$, there exists a function $G(x) \in L^1(-\infty, \infty)$ such that $g(t)$ is the Fourier transform of $G(x)$. Let $\{g_n(t)\}$ be a sequence of contractions of $f(t)$ such that $\lim_{n\to\infty} g_n(t) = 0$. Then, we only need for our purpose to show that

$$\lim_{n\to\infty} \|g_n\|_{\mathfrak{A}} = \lim_{n\to\infty} \int_{-\infty}^\infty | G_n(x) | dx = 0, \tag{1.1}$$

where each $G_n(x)$ is a function whose Fourier transform is $g_n(t)$. (This means that $f(t)$ is uniformly contractible in the space $\mathfrak{A}$.) For this purpose, we need the following inequality which was suggested by Professors S. and M. Izumi:

$$\int_0^\infty | G(x) | dx \leqq \int_0^\infty x^{-3/2}\left(\int_0^x u^2 | G(u) |^2 du\right)^{1/2} dx. \tag{1.2}$$

To prove the above inequality, we may suppose that $G(x) = 0$ for $x \geqq N > 0$. Putting $S(x) = \int_0^x u | G(u) | du$, we have

$$\int_0^\infty | G(x) | dx = \int_0^N | G(x) | dx = \int_0^N x^{-1} \frac{d}{dx} S(x)dx$$
$$= \int_0^N u^{-2} S(u)du + N^{-1} S(N).$$

Since, by Schwarz's inequality, $S(x) \leqq x^{1/2}(\int_0^x u^2 | G(u) |^2 du)^{1/2}$, we get

$$\int_0^\infty | G(x) | dx \leqq \int_0^N x^{-3/2}\left(\int_0^x u^2 | G(u) |^2 du\right)^{1/2} dx$$
$$+ 2N^{-1/2}\left(\int_0^N u^2 | G(u) |^2 du\right)^{1/2},$$

where the second part in the right-hand side is

$$\int_N^\infty x^{-3/2}dx\left(\int_0^N u^2|\,G(u)\,|^2du\right)^{1/2} \leqq \int_N^\infty x^{-3/2}\left(\int_0^x u^2|\,G(u)\,|^2du\right)^{1/2}dx.$$

Thus we get the inequality (1.2).

We now proceed to prove (1.1). We have

$$\int_0^x u^2|\,G_n(u)\,|^2du \leqq Cx^2\int_0^x |\,G_n(u)\,|^2 \sin^2\frac{u}{x}\,du$$

$$\leqq Cx^2\int_{-\infty}^\infty |\,G_n(u)\,|^2 \sin^2\frac{u}{x}\,du.$$

Then, by Parseval's relation, the right-hand side is equal to

$$(1.3)\qquad = Cx^2\int_{-\infty}^\infty |\,g_n(u+1/x) - g_n(u-1/x)\,|^2du$$

$$(1.4)\qquad \leqq Cx^2\int_{-\infty}^\infty |\,f(u+1/x) - f(u-1/x)\,|^2du$$

$$= Cx^2\int_{-\infty}^\infty |\,F(u)\,|^2 \sin^2\frac{u}{x}\,du$$

$$\leqq Cx^2\int_0^\infty w^2(u)\sin^2\frac{u}{x}\,du$$

$$(1.5)\qquad \leqq C\left\{\int_0^x u^2w^2(u)du + x^2\int_x^\infty w^2(u)du\right\}.$$

From the inequalities (1.2), (1.3), (1.4) and (1.5), we get

$$\|g_n\|_{\mathfrak{A}} \leqq C\int_{-\infty}^\infty dx\left\{|\,x\,|^{-1}\int_{-\infty}^\infty |\,g_n(u+1/x) - g_n(u-1/x)\,|^2du\right\}^{1/2}$$

$$\leqq C\int_{-\infty}^\infty dx\left\{|\,x\,|^{-1}\int_{-\infty}^\infty |\,f(u+1/x) - f(u-1/x)\,|^2du\right\}^{1/2}$$

$$\leqq C\int_0^\infty x^{-3/2}\left(\int_0^x u^2w^2(u)du\right)^{1/2}dx$$

$$+ C\int_0^\infty x^{-1/2}\left(\int_x^\infty w^2(u)du\right)^{1/2}dx$$

$$< \infty.$$

Note that $|\,g_n(u+1/x) - g_n(u-1/x)\,| \leqq |\,f(u+1/x) - f(u-1/x)\,|$; then, by the above inequality and by Lebesgue's convergence theorem, we see that

$$\lim_{n\to\infty} \|g_n\|_{\mathfrak{A}} = 0,$$

which completes the proof of Theorem 3.

2. We can discuss the spectral synthesis problem of bounded sequences analogously to the case in §1. In the case of bounded sequences, the space of absolutely convergent Fourier series, which we shall denote by A, plays the role of the space $\mathfrak{A}$ in the preceding section.

Suppose that $f(x)=\sum_{n=-\infty}^{\infty} c_n e^{inx}$ belongs to the space A, and define a norm of $f(x)$ in A by $\|f\|_A=\sum_{n=-\infty}^{\infty} |c_n|$. Let us suppose that a transformation $T(z)$ satisfies $T(0)=0$ and $|T(z)-T(z')| \leqq |z-z'|$; then we can define the terms "contractible" and "uniformly contractible" in the space A in a way similar to that used for the space $\mathfrak{A}$. (In this case, we omit the word "normalized" from the corresponding definitions.) Let a sequence $\{\phi_n\}$ be bounded and denote its spectral set by Sp.(ϕ_n). Then the theorems corresponding to Theorems 1 and 3 can be stated in the following way:

THEOREM 4. *Suppose that $f(x)\in A$ is uniformly contractible in the space A and $f(x)=\sum_{n=-\infty}^{\infty} c_n e^{inx}$ vanishes on* Sp.(ϕ_n). *Then we have $\sum_{n=-\infty}^{\infty} \phi_n c_n=0$.*

THEOREM 5. *Suppose that $\{w_n\}$ is a positive sequence such that $w_{-n}=w_n$ and satisfies the following condition*

$$\sum_{n=1}^{\infty} n^{-3/2}\left(\sum_{k=1}^{n} k^2 w_k^2\right)^{1/2} + \sum_{n=1}^{\infty} n^{-1/2}\left(\sum_{k=n+1}^{\infty} w_k^2\right)^{1/2} < \infty .$$

Then an element $f(x)$ of A is uniformly contractible in the space A when the Fourier coefficients $\{c_n\}$ of f satisfy $|c_n| \leqq w_{|n|}$.

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