STELLAR NEIGHBORHOODS IN POLYHEDRAL MANIFOLDS

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Introduction. It is the purpose of this paper to give some weaker analogues for polyhedral manifolds of well-known properties of combinatorial manifolds. This is now possible primarily because of the recent work of Mazur [**3**; **4**] and M. Brown [**1**]. A crucial issue in this area is the extent to which these analogues may be improved for arbitrary triangulated manifolds.

We shall prove a theorem which will be applied later on, after making some preliminary definitions. The join of two spaces X and Y is represented by $X \circ Y$. A map f of $X \circ Y - Y$ on itself is called *ray preserving* if for each ray $p \circ q - q$, p in X, q in Y, $f(p \circ q - q) \subseteq p \circ q - q$. A subset K of a cone $X \circ p$ is called *starlike* if $p \in K$ and each segment $p \circ x$, $x \in X$, meets K in a connected set; p is the *center* of K. Consider E^n to be the open cone over S^{n-1} from the origin p, in the usual way. We coordinatize E^n by (x, t), $x \in S^{n-1}$, t a real number with $0 \leqq t < \infty$; $(S^{n-1}, 0) = p$. Let D^n be the unit n-ball $p \circ (S^{n-1}, 1)$.

THEOREM 1. *Let K be a compact starlike set with center p, lying in the interior of D^n. There is a ray preserving map f of E^n on itself such that $f(K) = p$, $f|E^n - K$ is one-to-one and $f|E^n - D^n$ is the identity.*

PROOF. Let ρ be the usual euclidean metric for E^n. Choose $0 < \epsilon < 1$ so that $K \subseteq p \circ (S^{n-1}, \epsilon) = D_\epsilon$. First we define f on D_ϵ. If $z = (x, t) \in D_\epsilon$, $f(z) = (x, \rho(z, K))$. It is clear that f is a ray preserving map and monotone nonincreasing with respect to the coordinate t, so $f(D_\epsilon) \subseteq D_\epsilon$; furthermore $f(K) = p$ and $f|D_\epsilon - K$ is one-to-one. To see the last property suppose (x, t_1) and (x, t_2) are points of $D_\epsilon - K$ with $t_1 < t_2$. Let $y \in K$ so that $\rho((x, t_2), y) = \rho((x, t_2), K)$. Since $y \circ p \subseteq K$ and ρ is the ordinary straight line distance, obviously $\rho((x, t_1), y \circ p) < \rho((x, t_2), y)$ and accordingly $f(x, t_1) \neq f(x, t_2)$.

We next extend f to the annulus $D^n - D_\epsilon$ so that f is fixed on Bd D^n. If $z = (1-\tau) \cdot (x, \epsilon) + \tau \cdot (x, 1)$, $0 \leqq \tau \leqq 1$ and $x \in S^{n-1}$, let $f(z) = (1-\tau) \cdot f(x, \epsilon) + \tau \cdot (x, 1)$. Finally let f be fixed outside of D^n. Now f has the

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desired properties and is also monotone nonincreasing in the t-coordinate. Clearly f is realizable as the end of a ray preserving pseudo-isotopy starting at the identity, which is fixed outside of D^n and is monotone nonincreasing in t.

COROLLARY. *f is a homeomorphism of $E^n - K$ onto $E^n - p$.*

THEOREM 2. *Suppose X is a compact metric finite dimensional space and K is a compact starlike set in $X \circ p - X$. There is a ray preserving map f of $X \circ p$ on itself so that $f(K) = p$, f is a homeomorphism on $X \circ p - K$ and f is fixed on a neighborhood of X.*[2]

PROOF. Suppose X is imbedded in $(S^{n-1}, 1) \subseteq E^n$. This is extended in the obvious way to an imbedding of $X \circ p$ in D^n. K is now starlike in Int D^n so we apply Theorem 1 here. Since the map we then get is ray preserving and maps D^n on itself, clearly it maps $X \circ p$ on itself. The restriction of this map to $X \circ p$ satisfies the necessary conditions except possibly the last, however a slight alteration can be made in the proof of Theorem 1 so that the map constructed therein is also fixed on some neighborhood of Bd D^n.

1. A theorem due to Mazur.

In [3] Mazur indicated that Theorem 4 of this section would follow from the form of the Generalized Schoenflies Theorem later proved by M. Brown [1] and Morse [6]; he also gave a proof of Theorem 5 in [4]. In [7] we indicated a simpler proof of the latter, using Theorem 4. For completeness we include our own proofs.

THEOREM 3 (MOISE-ALEXANDER). *Let M be a compact Hausdorff space which is the union of two open sets, each of which is homeomorphic to E^n. Then M is homeomorphic to S^n.*

This was conjectured by Alexander for $n = 3$ and proved by Moise in that case [5]. Many people, including the author, have independently noticed that the proof for general n follows from [1] in a manner quite similar to Moise's proof.

THEOREM 4 (MAZUR). *If $Y \circ p$ is locally n-euclidean at p, then the suspension $S(Y)$ is homeomorphic to S^n.*

PROOF. Consider $S(Y)$ as $Y \times [-1, 1]$ with Y_{-1} and Y_1 identified as distinct points p' and p, respectively. Suppose U is a euclidean neighborhood of p in $S(Y)$ so that $\overline{U}$ is compact. For some t_0, $-1 < t_0 < 1$, $\bigcup(Y_t: t_0 \leqq t \leqq 1) \subseteq U$. Y_{t_0} is closed in $S(Y)$, so Y, hence

[2] A more general theorem can be proved but this is sufficient for our applications.

$S(Y)$, is compact. Let h be a homeomorphism of $S(Y)$ on itself, fixed on Y_{t_0}, which takes $\bigcup(Y_t: t_0 \leqq t)$ onto $\bigcup(Y_t: t \leqq t_0)$. U and $h(U)$ are two euclidean neighborhoods which cover $S(Y)$.

COROLLARY. *$Y \circ p - Y$ is homeomorphic to E^n.*

PROOF. There is a simple homeomorphism of $Y \circ p - Y$ onto $\bigcup(Y_t: 0 < t)$ in $S(Y)$ and a homeomorphism of the second set onto $S(Y) - p'$.

THEOREM 5 (MAZUR). *Let v be a vertex of a triangulated n-manifold M. The open star of v in M is homeomorphic to E^n.*

PROOF. Let B be the link of v in M. Since $B \circ v - B$ is a neighborhood of v in M, by Theorem 4 $S(B) \approx S^n$. (We use the notation $X \approx Y$ if X and Y are homeomorphic spaces.) Therefore by the corollary, $B \circ v - B \approx E^n$.

2. **Some analogues to combinatorial theorems.** M. Brown has called a subset of an n-manifold *cellular* if it is the intersection of a sequence (C_n) of topological n-cells where $C_{n+1} \subseteq \operatorname{Int} C_n$. Let Q be a subpolyhedron of a polyhedron P. The *stellar neighborhood of Q in P* is the union of all open simplexes of P which have vertices lying in Q. *Star* Q is the union of all closed simplexes of P which meet Q. Q is said to be *full* in P if it contains each simplex of P, all of whose vertices lie in Q.

Henceforth all simplexes will be considered closed. If σ is a simplex of P, by σ^* we mean the union of all simplexes of P which contain σ, or equivalently $\sigma^* = \sigma \circ \operatorname{Lk} \sigma$—where $\operatorname{Lk} \sigma$ is the link of σ in P; σ^* is called the *cluster* of σ. $\dot{\sigma}$ is the union of all proper faces of σ; $\sigma^b = \sigma - \dot{\sigma}$. Finally $\beta(\sigma)$ will be the barycenter of σ.

THEOREM 6. *Let K be a full finite subpolyhedron of a polyhedral n-manifold M. Then K is cellular in M if and only if its stellar neighborhood is homeomorphic to E^n.*

PROOF. (a) Assume K is cellular. Let V be the stellar neighborhood of K, $B = \operatorname{Bd} V$ and $N = \operatorname{Star} K = \bar{V}$. Clearly $B \neq \emptyset$. Let X be the set of all midpoints of straight segments joining points of B to points of K. We shall show that the decomposition space of V with K identified to a point is homeomorphic to the open cone over X. For since K is full, each simplex of N can be represented as a join $\sigma \circ \tau$, $\sigma \subseteq K$, $\tau \subseteq B$. When K is identified to a point, $\sigma \circ \tau - \sigma$ becomes simply the open cone over the set of all midpoints of segments from τ to σ; the vertex of the cone is the image of K in the decomposition space. Since such a representation holds for each maximal simplex in N, the assertion is easily established.

Now let C be a topological n-cell lying in V with $K \subseteq \text{Int } C$. By Theorem 1 of [1] there is a map f of V on itself fixed outside of C such that $f(K)$ is a point and $f| V-K$ is one-to-one. Hence the existence of the map f shows that V is homeomorphic to the open cone over X, with vertex $f(K)$. By Theorem 4 and its corollary, $V \approx E^n$.

(b) Now suppose $V \approx E^n$. Since K is full we can conclude from the argument in part (a) that V can be represented as an open mapping cylinder over K (from the space X) using the segments from B to K minus their endpoints in B. Since K is compact there is a topological n-cell C_1 in V so that $K \subseteq \text{Int } C_1$. Now let each segment be linearly parametrized from 0 to 1 with 0 the parameter of the endpoint in K. There is an ϵ_1, $0 < \epsilon_1 < 1/2$, so that on each segment the subsegment from 0 to ϵ_1 lies in Int C_1. Hence let h_1 be a homeomorphism of V into Int C_1, fixed on K, which maps each ray from 0 to 1 linearly onto its subray from 0 to ϵ_1. Thus there is a topological n-cell C_2 in Int C_1 with $K \subseteq \text{Int } C_2$. The process may be continued by induction to define a sequence (C_n) of n-cells having the properties that $\bigcap_n C_n = K$ and $C_{n+1} \subseteq \text{Int } C_n$.

COROLLARY. *Let K be a finite subpolyhedron of a polyhedral n-manifold M. Then K is cellular if and only if its first barycentric stellar neighborhood in M is homeomorphic to E^n.*

PROOF. The first barycentric subdivision of K is full in the corresponding subdivision of M by Lemma 9.4, p. 71 of [2].

It may be noted that Theorem 6 generalizes Theorem 5 since vertices are clearly cellular.

THEOREM 7. *Let σ be a simplex in a polyhedral n-manifold M. Then* Int σ^* *is homeomorphic to E^n.*

PROOF (*Added in proof*). Adapting the notation of Alexander in [8] (our manifolds may be noncompact) let $M = \sigma \circ \text{Lk } \sigma + R$ and consider the simple subdivision $M \to \beta(\sigma) \circ (\dot{\sigma} \circ \text{Lk } \sigma) + R$. By Theorem 5, E^n is homeomorphic to $\text{Int}(\beta(\sigma) \circ (\dot{\sigma} \circ \text{Lk } \sigma)) = \text{Int } \sigma^*$. This method also tells us that, by Theorem 4, $S(\dot{\sigma} \circ \text{Lk } \sigma) \approx S^n$. Hence we have the

COROLLARY (*Added in proof*). *If σ is a k-dimensional simplex in a triangulated n-manifold then* $S^k \circ \text{Lk } \sigma \approx S^n$.

PROOF (*Added in proof*). For $S(\dot{\sigma} \circ \text{Lk } \sigma) = S(\dot{\sigma}) \circ \text{Lk } \sigma$.

THEOREM 8. *Let M be an n-manifold with triangulation T. Let σ be a simplex of the first barycentric subdivision T' of T. Then σ is cellular in M.*

PROOF. Let τ be the lowest dimensional simplex of T for which $\beta(\tau)$ is a vertex of σ. Obviously $\tau^b \subseteq \operatorname{Int} \sigma^*$. We shall show that Star $\sigma \subseteq \tau^*$, Star σ formed with respect to T' and τ^*, with respect to T. For any other vertex $\beta(\tau')$ of σ, $\tau' \supseteq \tau$; so if τ^n is an n-dimensional simplex containing $\beta(\tau')$, we have $\tau^n \supseteq \tau$ and hence $\tau^n \subseteq \tau^*$. It now may easily be seen that $\sigma \subseteq \operatorname{Int} \tau^*$.

Again consider the simple subdivision $M = \tau \circ \operatorname{Lk} \tau + R \rightarrow \beta(\tau) \circ (\dot{\tau} \circ \operatorname{Lk} \tau) + R$. We can see that σ is even a subcone, thus a compact starlike set in $\operatorname{Int}(\beta(\tau) \circ (\dot{\tau} \circ \operatorname{Lk} \tau)) = \operatorname{Int} \tau^*$. Since $\operatorname{Int} \tau^* \approx E^n$ and by Theorem 2 σ is pointlike in $\operatorname{Int} \tau^*$, it follows by Theorem 3 of [1] that σ is cellular.

COROLLARY. *The stellar neighborhood of σ with respect to T' is homeomorphic to E^n.*

PROOF. σ is full in T'.

THEOREM 9 (*Added in proof*). *Let M be an n-manifold with triangulation T. If σ is a simplex of the second barycentric subdivision T'' of T then σ^* is cellular in M.*

PROOF (*Added in proof*). T' will denote the first barycentric subdivision of T. Suppose σ is a simplex of T''. Let τ be the lowest dimensional simplex of T' whose barycenter $\beta(\tau)$ is a vertex of σ. As before, it follows that $\sigma^* \subseteq \tau^*$. It is also fairly easy to see that each vertex of τ lies in σ^*.

Now let υ be the minimal simplex of T whose barycenter $\beta(\upsilon)$ is a vertex of τ. We may conclude that υ is the minimal simplex of T whose barycenter lies in σ^*. For if $\upsilon_1 \in T$ and $\beta(\upsilon_1) \in \sigma^*$, denote by σ_1 the simplex of T'' which contains σ and $\beta(\upsilon_1)$. Since σ_1 must be n-dimensional let τ_1 be the n-simplex of T' containing σ_1. Inasmuch as $\sigma_1 \supseteq \sigma$, $\tau_1 \supseteq \tau$ and $\beta(\tau) \in \sigma$, it may be seen that $\beta(\upsilon_1) \in \tau$. This proves that $\upsilon \subseteq \upsilon_1$.

Consider the simple subdivision $M = \upsilon \circ \operatorname{Lk} \upsilon + R \rightarrow \beta(\upsilon) \circ P + R$, where $P = \dot{\upsilon} \circ \operatorname{Lk} \upsilon$. Evidently $\sigma^* \subseteq \beta(\upsilon) \circ P$; we shall prove that $\sigma^* \subseteq \operatorname{Int} \beta(\upsilon) \circ P$, the latter, of course, being a homeomorph of E^n. Firstly, it may be established without too much difficulty, by a straightforward induction on n, that if $\tau^n \subseteq \beta(\upsilon)^*$ with respect to T' then $\tau^n \cap P \subseteq \dot{\upsilon}$. Now suppose $\sigma^n \in T''$ so that $\sigma^n \supseteq \sigma$ and $\sigma^n \cap \dot{\upsilon} \neq \emptyset$. Let $\tau^n \in T'$ with $\sigma^n \subseteq \tau^n$ and $\tau_1 = \tau^n \cap \dot{\upsilon}$. Then σ^n contains $\beta(\tau_0)$ for some face τ_0 of τ_1; this in turn necessitates that σ^n contains a vertex v of τ_0. The vertex v must be the barycenter of a face v_0 of $\dot{v}$. Since $v\in\sigma^*$, this contradicts the minimality condition on v proved above.

Finally we show that σ^* is starlike in $\beta(v) \circ P$. It will then be obvious from our previous arguments that σ^* is cellular in M. Let τ^n be any n-simplex of T' which contains σ. By elementary analytic geometry it may be verified that $\sigma^*\cap\tau^n$ is convex in τ^n. (See for example formula (2) on p. 62 of [2].) This completes the proof of the theorem.

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