ON THE BOUNDARY VALUES OF BLASCHKE PRODUCTS AND THEIR QUOTIENTS

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1. **Introduction.** Let $B(z)$ be the infinite Blaschke product:

$$e^{i\lambda}\, z^m \prod_{n=1}^{+\infty} \bar{a}_n(a_n - z)/|a_n|(1 - \bar{a}_n z),$$

where λ is a real constant, and m is a non-negative integer, $0<|a_n|<1$, $\sum_{n=1}^{+\infty}(1-|a_n|)<+\infty$. The object of this note is to establish the following two theorems.

THEOREM 1. (A) *If the subsequence* $\{a_{n_k}\}$ *tends to* $z=e^{i\phi}$ *within the Stolz domain in such a manner that*

$$\lim_{k\to+\infty} |(a_{n_k} - a_{n_{k+1}})|/|a_{n_k} - e^{i\phi}| = 0,$$

then the angular limit at $e^{i\phi}$ *of* $B(z)$ *is* 0.

(B) *If the subsequence* $\{a_{n_k}\}$ *tends to* $z=e^{i\phi}$ *within the circle*: $|z-ae^{i\phi}|\leqq 1-a$ $(0<a<1)$, *in such a manner that*

$$\lim_{k\to+\infty} 1/x_k^2\cdot |a_{n_k} - a_{n_{k+1}}| = 0,$$

where $x_k=\min\{|a_{n_k}-e^{i\phi}|, |a_{n_{k+1}}-e^{i\phi}|\}$, *then the angular limit at* $e^{i\phi}$ *of* $B(z)$ *is* 0.

As an application of Theorem 1 (A), we prove

THEOREM 2. *There exists a meromorphic function* $f(z)$ *of bounded characteristic in* $|z|<1$ *represented by the quotient of two infinite Blaschke products such that*

(1) $f(z)$ *has infinite number of zeros and poles on* $\arg(1-z)=-\vartheta$ *and* $\arg(1-z)=+\vartheta$ *respectively* $(0<\vartheta<\pi/2)$.

(2) $$\lim_{z\to1;\, \arg(1-z)=-\vartheta} f(z) = 0, \quad \textit{and} \quad \lim_{z\to1;\, \arg(1-z)=+\vartheta} f(z) = \infty.$$

REMARK. (1) O. Frostman [1, p. 109] was the first to construct an example of Blaschke product with the boundary value 0, i.e.,

$$B(z) = \prod_{n=1}^{+\infty}\{(1-1/n^2)-z\}/\{1-(1-1/n^2)z\}, \quad \text{where } \lim_{r\to1} B(r) = 0.$$

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(2) By the well-known Iversen-Lindelöf theorem on asymptotic values, $f(z)$ of Theorem 2 has Picard's property in the sector S: $|\arg(1-z)| \leqq \vartheta < \pi/2$; $w=f(z)$ takes every value w, except perhaps two, infinitely many times in S. On the other hand, $f(e^{i\theta})$ is of modulus one almost everywhere on $|z|=1$.

(3) D. A. Storvick [3, p. 37] constructed a meromorphic function $f(z)$ defined by the quotient of two infinite Blaschke products such that $f(z)\to 0$ and ∞ as $z\to 1$ along the upper and lower oricycle: $r=\cos\theta$ respectively, $z=re^{i\theta}$.

2. **Proof of Theorem 1.** (A) We decompose $B(z)$ as follows:

$$B(z) = B_1(z)\cdot B_2(z),$$

where $B_1(z) = \prod_{k=1}^{+\infty} \bar{a}_{n_k}(a_{n_k}-z)/|a_{n_k}|(1-\bar{a}_{n_k}z)$, $B_2(z)=B(z)/B_1(z)$. Since $|B(z)|<|B_1(z)|$ for $|z|<1$, it is sufficient to prove that the angular limit at $e^{i\phi}$ of $B_1(z)$ is 0.

Without any loss of generality, we can assume that $\phi=0$. Put $z=1-re^{i\theta}$, $a_{n_k}=b_k=1-r_ke^{i\theta_k}$. By a simple calculation,

$$\begin{aligned}(2.1)\quad &(b_k - z)/(1-\bar{b}_kz)\\ &= (b_k-z)/r_ke^{-i\theta_k}\cdot\{(e^{i2\theta_k}+1) - re^{i\theta} + (b_k-z)/r_ke^{i\theta_k}\cdot e^{i2\theta_k}\}^{-1}.\end{aligned}$$

Let us denote by l_k the segment connecting two points b_k and b_{k+1}. If z lies on l_k, we have evidently

$$(2.2)\qquad |b_k - z| \leqq |b_k - b_{k+1}|, \qquad r \leqq \max(r_k, r_{k+1}).$$

By $|\theta_k| \leqq \vartheta < \pi/2$, we get easily

$$(2.3)\qquad |e^{i2\theta_k}+1| > \sin(2\vartheta).^{1}$$

By (2.1), (2.2) and (2.3)

$$\begin{aligned}|(b_k - z)/(1-\bar{b}_kz)| \leqq\ & |(b_k-b_{k+1})|/|1-b_k|\\ &\cdot\{\sin(2\vartheta) - \max(r_k, r_{k+1}) - |(b_k-b_{k+1})|/|1-b_k|\}^{-1},\end{aligned}$$

so that, by the assumptions:

$$\lim_{k\to\infty} |(b_k-b_{k+1})|/|1-b_k| = 0, \qquad \lim_{k\to\infty}\max(r_k, r_{k+1}) = 0,$$

we obtain

$$(2.4)\qquad \lim_{k\to\infty} (b_k - z)/(1-\bar{b}_kz) = 0,$$

where $z\in l_k$. Since

[1] $|e^{i2\theta_k}+1| = 2\cos\theta_k \geqq 2\cos\vartheta > \sin(2\vartheta)$.

$$|B_1(z)| < |(b_k - z)/(1 - \bar{b}_k z)| \qquad \text{for any } k \text{ and } |z| < 1,$$

by (2.4)

$$\lim B_1(z) = 0,$$

as $z \to 1$ along $C = \bigcup_k l_k$. Hence, by Lindelöf's theorem [2, p. 5]

$$\lim_{z \to 1} B_1(z) = 0,$$

as $z \to 1$ inside a Stolz domain with vertex at $z = 1$, as was to be proved.

(B) Using the same notations as above, we get

(2.5) $(b_k - z)/(1 - \bar{b}_k z) = (b_k - z)/rr_k e^{i(\theta - \theta_k)} \cdot \{e^{i\theta_k}/r_k + e^{-i\theta}/r - 1\}^{-1}$.

In the circle: $|z - a| \leqq 1 - a$ $(0 < a < 1,\ z = 1 - re^{i\theta})$, we have

$$\frac{1}{2(1 - a)} \leqq \cos\theta/r. \tag{2.6}$$

If z lies on l_k, by (2.5) and (2.6),

$$\begin{aligned} |(b_k - z)/(1 - \bar{b}_k z)| &\leqq |(b_k - z)|/rr_k \cdot \{\cos\theta_k/r_k + \cos\theta/r - 1\}^{-1} \\ &\leqq (1/a - 1) \cdot y_k/(\min(r) \cdot x_k), \end{aligned}$$

where $y_k = |b_k - b_{k+1}|$, $x_k = \min(r_k, r_{k+1})$, $\min(r) = \min_{z \in l_k} |z - 1|$. If $\min(r) = x_k$, we have

$$|(b_k - z)/(1 - \bar{b}_k z)| \leqq (1/a - 1) \cdot y_k/x_k^2.$$

If $\min(r) < x_k$, we have easily

$$\min(r) \geqq (x_k^2 - (y_k/2)^2)^{1/2},$$

so that

$$|(b_k - z)/(1 - \bar{b}_k z)| \leqq (1/a - 1) \cdot y_k/x_k^2 \cdot \{1 - (y_k/2x_k)^2\}^{-1/2}.$$

In any case, by the assumption: $\lim_{k \to \infty} y_k/x_k^2 = 0$, we have

$$\lim_{k \to \infty} (b_k - z)/(1 - \bar{b}_k z) = 0,$$

as z on l_k. Hence, by entirely similar arguments as in (A),

$$\lim_{z \to 1} B(z) = 0,$$

as $z \to 1$ inside the Stolz domain with vertex at $z = 1$.

3. **Lemmas.** To prove Theorem 2, we need two lemmas.

LEMMA 1. *Put*

$$w(z) = (a - z)(1 - az)/(1 - \bar{a}z)(\bar{a} - z),$$

where $|a| < 1$, $I(a) > 0$.[2] *Then*

$$|w(z)| < 1 \qquad \textit{for } |z| < 1,\ I(z) > 0.$$

PROOF. $w(z)$ is regular in the upper semi-circle D: $|z| \leqq 1$, $I(z) \geqq 0$. On the boundary of D, we have evidently $|w| = 1$. Hence, by the maximum-modulus principle, $|w(z)| < 1$ for $|z| < 1$, $I(z) > 0$.

LEMMA 2. *In the domain* D: $|z| < 1$, $I(z) \geqq 0$, $|z-1| \leqq |a-1|$, *where* $|a| < 1$, $I(a) > 0$, *we have*

$$|(1 - az)/(z - \bar{a})| < \exp(2/\sin^2 \vartheta),$$

where $\arg(1-a) = -\vartheta$ $(0 < \vartheta < \pi/2)$.

PROOF. By the inequality: $\log(1+x) \leqq x$ for $x \geqq 0$, for $|a| < 1$, $|z| \leqq 1$ we obtain

$$\begin{aligned} \log |(1 - az)/(z - \bar{a})| &= \frac{1}{2}\log\{1 + (1 - |a|^2)(1 - |z|^2)/|(z - \bar{a})|^2\} \\ &\leqq \frac{1}{2}(1 - |a|^2)(1 - |z|^2)/|(z - \bar{a})|^2 \\ &< 2|(1 - a)(1 - z)|/|(z - \bar{a})|^2. \end{aligned}$$

Hence

$$\log |(1 - az)/(z - \bar{a})| < 2|1 - a|^2/|I(a)|^2 = 2/\sin^2\vartheta \qquad \text{for } z \in D,$$

because $|z-1| \leqq |a-1|$, $|z-\bar{a}| \geqq I(a)$ in D. Thus Lemma 2 is proved.

4. **Proof of Theorem 2.** Let the sequence $\{\epsilon_n\}$ be such that

$$\cos\vartheta > \epsilon_1 > \epsilon_2 > \cdots > \epsilon_n > \to 0,$$

$$\sum_{n=1}^{+\infty} \epsilon_n < +\infty, \tag{4.1}$$

$$\lim_{n\to\infty} \epsilon_{n+1}/\epsilon_n = 1.$$

Put $a_n = 1 - \epsilon_n \cdot e^{-i\vartheta}$ $(0 < \vartheta < \pi/2)$. Then

$$|a_n| < 1, \qquad I(a_n) > 0 \qquad \text{for } n \geqq 1.$$

The desired function $f(z)$ is given by $f(z) = B_1(z)/B_2(z)$, where

[2] $I(a)$ is the imaginary part of a.

$$B_1(z) = \prod_{n=1}^{+\infty} \bar{a}_n(a_n - z)/\,|\,a_n\,|\,(1 - \bar{a}_n z),$$

$$B_2(z) = \prod_{n=1}^{+\infty} a_n(\bar{a}_n - z)/\,|\,a_n\,|\,(1 - a_n z).$$

Since $\sum_{n=1}^{+\infty} 1-|a_n| < \sum_{n=1}^{+\infty} |1-a_n| = \sum_{n=1}^{+\infty} \epsilon_n < +\infty$, the Blaschke products $B_i(z)$ $(i=1, 2)$ are convergent.

We can put

$$f(z) = \prod_{n=1}^{+\infty} \bar{a}_n/a_n\cdot(a_n - z)(1 - a_n z)/(1 - \bar{a}_n z)(\bar{a}_n - z),$$

so that, by Lemmas 1 and 2, we have

$$|f(z)| < |(a_k - z)/(1 - \bar{a}_k z)| \cdot \exp(2/\sin^2\vartheta) \tag{4.2}$$

on the segment: $\arg(1-z) = -\vartheta$, $|1-z| \leqq \epsilon_k$. By (4.1)

$$|(a_k - a_{k+1})/(1 - a_k)| = 1 - \epsilon_{k+1}/\epsilon_k \to 0 \qquad \text{as } k \to +\infty.$$

Hence, by (4.2) and arguments similar to those in the proof of Theorem 1 (A),

$$\lim_{z\to 1;\, \arg(1-z) = -\vartheta} f(z) = 0.$$

Similarly

$$\lim_{z\to 1;\, \arg(1-z) = +\vartheta} 1/f(z) = 0.$$

Thus Theorem 2 is completely established.

References

1. O. Frostman, *Potentiel d'équilibre et capacité des ensembles*, Thèse, Lund, 1935.

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3. D. A. Storvick, *On meromorphic functions of bounded characteristic*, Proc. Amer. Math. Soc. **8** (1957), 32–38.

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