

$F \cdot (C_1 \cdot C_2) \neq \emptyset$. Choose p an element of the set $F \cdot (C_1 \cdot C_2)$. Now since $F \subset C_\beta$ for every β , it follows that p is an element of their intersection.

In the case that the union of no two sets separates the plane it follows that the intersection of any two sets is connected. The proof of the theorem is then completed by the use of Theorem 1.

REFERENCES

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A NOTE ON EXACT SEQUENCES

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1. We consider a commutative diagram of abelian groups with exact rows:

$$\begin{array}{ccccccc} A_1 & \rightarrow & A_2 & \xrightarrow{\beta} & A_3 & \xrightarrow{\gamma} & A_4 \rightarrow A_5 \\ \downarrow \phi_1 & & \downarrow \phi_2 & & & & \downarrow \phi_4 \quad \downarrow \phi_5 \\ B_1 & \rightarrow & B_2 & \rightarrow & B_3 & \rightarrow & B_4 \rightarrow B_5 \end{array}$$

FIGURE 1

We suppose the ϕ_i are isomorphisms onto. The five lemma states that if $\phi_3: A_3 \rightarrow B_3$ is a homomorphism for which the diagram (with ϕ_3 inserted) commutes, then ϕ_3 is an isomorphism onto.

Suppose $\eta_1, \eta_2: A_3 \rightarrow B_3$ both give commutative diagrams, so that each is an isomorphism onto. (It is easy to find examples in which $\eta_1 \neq \eta_2$.)

PROPOSITION I. *Under the above hypotheses,*

$$\eta_1^{-1}\eta_2(x) + \eta_2^{-1}\eta_1(x) = 2x$$

for all $x \in A_3$.

To prove this, let $f = \eta_1^{-1}\eta_2$, and let I be the identity function on A_3 .

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The equation reads $f + f^{-1} = 2I$, which is easily seen to be equivalent to $(f - I)(f - I) = 0$. In fact, applying f to $f + f^{-1} = 2I$, we get $(f - I)^2 = 0$. In proving the reverse implication, the only question is whether f has an inverse. However, f is onto, for if $y \in A_3$, then $f(y - (f - I)y) = y$. Also f is 1-1, for if $f(x) = 0$, then $(f - I)(x) = -x$, and applying $(f - I)$ again, we see that $0 = (f - I)(-x) = x$. To see that $(f - I)^2 = 0$, we observe that $(f - I)\beta = 0$ and that $\gamma(f - I) = 0$. Thus $\text{image } (f - I) \subset \text{kernel } \gamma = \text{image } \beta$, and $(f - I)^2 = 0$.

Note that the extremities of the diagram were not used in the proof, except to insure that η_1 and η_2 were isomorphisms onto. In fact, we may reason to the same conclusion from the commutative diagram

$$\begin{array}{ccccc} & \beta & & \gamma & \\ A_2 & \rightarrow & A_3 & \rightarrow & A_4 \\ \downarrow & \eta_1 \downarrow & \downarrow \eta_2 & \downarrow \phi_4 & \\ B_2 & \rightarrow & B_3 & \rightarrow & B_4 \end{array}$$

where η_1 and η_2 are isomorphisms onto, ϕ_4 is 1-1, and the rows are exact.

Now suppose (in either diagram) that A_3 and B_3 are rings, and $\eta_1: A_3 \rightarrow B_3$ is a ring isomorphism onto making the diagram commute. We consider the question of whether there is another ring isomorphism $\eta_2: A_3 \rightarrow B_3$ making the diagram commute.

PROPOSITION II. *A necessary and sufficient condition for the existence of a ring isomorphism $\eta_2: A_3 \rightarrow B_3$ making the diagram commute, with $\eta_2 \neq \eta_1$, is that there exist a nontrivial additive homomorphism $\delta: A_3 \rightarrow A_3$ such that*

- (i) $\delta\beta = 0$, $\gamma\delta = 0$ (consequently, $\delta^2 = 0$) and
- (ii) $\delta(xy) = (\delta x)y + x\delta y + (\delta x)(\delta y)$ for all $x, y \in A_3$.

PROOF. If such a δ exists, define $\eta_2 = \eta_1 + \eta_1\delta$. It is trivial that η_2 is a ring homomorphism, that the diagram commutes, and that η_2 is 1-1. To verify that η_2 is onto, let $y \in B_3$, and let $\eta_1(x) = y$. Then $\eta_2(x - \delta x) = y$. On the other hand, suppose η_1 and $\eta_2 \neq \eta_1$ are given. Let $\delta = \eta_1^{-1}\eta_2 - I$. Then (i) and (ii) are easily verified.

REMARKS. Applying δ to (ii), we get

$$2(\delta x)(\delta y) = 0 \quad \text{for all } x, y \in A_3.$$

Since $\text{image } \delta \subset \text{image } \beta$, in order for δ to be nontrivial it is necessary that $\text{image } \beta$ contain divisors of zero or elements of order two.

If A_3 has a two-sided identity element 1, then it follows that $\delta(1) = 0$.

An example in which $\eta_1 \neq \eta_2$ is given by Figure 1, where the two rows are identical, $A_1 = A_5 = \{0\}$, $A_2 = A_4 = Z_2$ (the integers modulo two), $A_3 = B_3 = Z_2 \times Z_2$, $\phi_i = \text{identity} = \eta_1, \eta_2(x, y) = (y, x)$, $\beta(x) = (x, x)$, and $\gamma(x, y) = x + y$, for $x, y \in Z_2$. η_1 and η_2 are both ring isomorphisms.

2. An application. We consider $(k-1)$ -sphere bundles $\mathfrak{B}' = (E', p', B', S^{k-1}, SO(k))$ and $\mathfrak{B} = (E, p, B, S^{k-1}, SO(k))$ with group the special orthogonal group $SO(k)$; and two bundle maps $f, g: \mathfrak{B}' \rightarrow \mathfrak{B}$. Let \tilde{f}, \tilde{g} be the mappings of the base spaces induced by f and g , respectively. We use a field F of characteristic not two for coefficients for cohomology, and we assume that $\tilde{g}^* = \tilde{f}^*: H^*(B, F) \rightarrow H^*(B', F)$ and that this is an isomorphism onto. Then if $H^*(E, F)$ has no divisors of zero, $f^* = g^*: H^*(E, F) \rightarrow H^*(E', F)$. To see this, apply the remark following Proposition II to the isomorphisms of the Gysin sequences induced by f and g :

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^{q-k}(B) & \longrightarrow & H^q(B) & \longrightarrow & H^q(E) \longrightarrow H^{q-k+1}(B) \longrightarrow \cdots \\ & & \tilde{f}^* = \tilde{g}^* \downarrow & & \tilde{f}^* = \tilde{g}^* \downarrow & & f^* \downarrow \downarrow g^* \quad \quad \downarrow \tilde{f}^* = \tilde{g}^* \\ \cdots & \longrightarrow & H^{q-k}(B') & \longrightarrow & H^q(B') & \longrightarrow & H^q(E') \longrightarrow H^{q-k+1}(B') \longrightarrow \cdots \end{array}$$

An example in which $H^*(E, F)$ has no divisors of zero is furnished by taking \mathfrak{B} to be the universal $(k-1)$ -sphere bundle $(B_{k-1}, p, B_k, S^{k-1}, SO(k))$, where B_n is the classifying space for $SO(n)$. For the cohomology of B_n see A. Borel, *Topology of Lie groups and characteristic classes*, Bull. Amer. Math. Soc. 61 (1955), 397-432.

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