CONCERNING AN APPROXIMATION OF COPSON

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For each complex number z and each positive integer n, define $S_n(z)$ by the equation

$$e^{nz} = \sum_{p=0}^{n} \frac{(nz)^p}{p!} + \frac{(nz)^n}{n!} S_n(z). \tag{1}$$

In 1913, Ramanujan [2] made the assertion (in a somewhat different notation) that

$$S_n(1) = \frac{n!}{2}\left(\frac{e}{n}\right)^n - \frac{2}{3} + \frac{4}{135n} + O\left(\frac{1}{n^2}\right).$$

Proofs for this and related matters were given independently in 1928 by G. Szegö [3] and G. N. Watson [4].

In 1932, E. T. Copson [1] proved that $\{S_n(-1)\}$ is a decreasing sequence with limit $-\frac{1}{2}$, and derived the asymptotic series

$$S_n(-1) \sim -\frac{1}{2} + \frac{1}{8n} + \frac{1}{32n^2} - \frac{1}{128n^3} - \frac{13}{256n^4} + \cdots.$$

The determination of the coefficients is quite complicated (and, for the coefficient of n^{-4}, incorrect).

In the present paper we obtain Copson's series by a simpler method which yields an asymptotic expansion for $S_n(z)$ valid for every complex number z except $z=1$.

For our principle result we require the following three lemmas concerning the function $S_n(z)$ defined by (1) and the function $T_n(z)$ defined by

$$e^{nz} = \frac{(nz)^n}{n!} T_n(z) + \sum_{p=n+1}^{\infty} \frac{(nz)^p}{p!}. \tag{2}$$

LEMMA 1. *For each positive integer n, the functions $S_n(z)$ and $T_n(z)$ have the following properties*:

$$S_n(z) + T_n(z) = \frac{n!(e/n)^n}{(ze^{1-z})^n}; \tag{i}$$

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(ii) $$S_n(z) = \frac{z}{1-z}\left[1 - \frac{S_n'(z)}{n}\right]$$

and

$$T_n(z) = \frac{z}{z-1}\left[1 + \frac{T_n'(z)}{n}\right];$$

(iii) $|S_n(z)| < en^{1/2}$ *if* $|z| \leqq 1$, $|T_n(z)| < en^{1/2}$ *if* $|z| \geqq 1$; *and if* $|ze^{1-z}| \geqq 1$, *then both* $|S_n(z)|$ *and* $|T_n(z)|$ *are less than* $2en^{1/2}$.

Indication of proof. (i) and (ii) are direct consequences of the definitions. To prove (iii) we make use of the fact that

$$\begin{aligned} |S_n(z)| &\leqq S_n(1) \quad \text{if } |z| \leqq 1, \text{ and} \\ |T_n(z)| &\leqq T_n(1) \quad \text{if } |z| \geqq 1. \end{aligned}$$

Since each of $S_n(1)$ and $T_n(1)$ is positive, neither is as large as $S_n(1)+T_n(1)=n!(e/n)^n \leqq en^{1/2}$; the last part of (iii) then follows from (i).

Notation. The linear operator which transforms $f(z)$ into $zf'(z)/(z-1)$ will be denoted by $(z/(z-1))(d/dz)$. J and K will be the sets given by $\{z: |z| \leqq 1 \text{ and } |ze^{1-z}| \leqq 1\}$ and $\{z: |z| \geqq 1 \text{ and } |ze^{1-z}| \leqq 1\}$, respectively. If M is a set, $d_M(z)$ and $\phi_M(z)$ will denote the *distance* and *characteristic* functions for the set M; specifically, $d_M(z)$ is the greatest lower bound of distances from z to points of M, and $\phi_M(z)$ is 1 or 0 according as z is or is not in M.

Lemma 2. *For every positive integer k,*

$$S_n(z) = \sum_{r=0}^{k-1}\left(\frac{1}{n}\frac{z}{z-1}\frac{d}{dz}\right)^r \frac{z}{1-z} + \left(\frac{1}{n}\frac{z}{z-1}\frac{d}{dz}\right)^k S_n(z),$$

and

$$T_n(z) = \sum_{r=0}^{k-1}\left(\frac{1}{n}\frac{z}{z-1}\frac{d}{dz}\right)^r \frac{z}{z-1} + \left(\frac{1}{n}\frac{z}{z-1}\frac{d}{dz}\right)^k T_n(z).$$

Proof. Mathematical induction and (ii) of Lemma 1.

Lemma 3. *Suppose* $0<\epsilon<1$. *If k is a nonnegative integer, then*

(i) $$\left|\left(\frac{z}{z-1}\frac{d}{dz}\right)^k S_n(z)\right| < 14\,\frac{2^{k^2+3k}}{\epsilon^{2k+2}} \qquad \text{if } d_K(z) \geqq \epsilon,$$

and

$$\text{(ii)} \qquad \left|\left(\frac{z}{z-1}\frac{d}{dz}\right)^k T_n(z)\right| < 14\,\frac{2^{k^2+3k}}{\epsilon^{2k+2}} \qquad \textit{if } d_J(z) \geqq \epsilon.$$

PROOF. Since the proofs for the two inequalities are essentially the same, we shall omit the proof of (ii).

Suppose first that $k=0$. From Lemma 1, $|S_n(z)| < 2en^{1/2}$ for every point z in the closure of the complement of K. Therefore, if $d_K(z) \geqq \epsilon$,

$$|S_n'(z)| < \frac{2en^{1/2}}{d_K(z)} \leqq \frac{2en^{1/2}}{\epsilon}$$

by the Cauchy inequality for derivatives. Since 1 is in K, $|z-1| \geqq \epsilon$. Therefore, from (ii) of Lemma 1, we have

$$|S_n(z)| < \left(1+\frac{1}{\epsilon}\right)\left(1+\frac{2en^{-1/2}}{\epsilon}\right) < \frac{14}{\epsilon^2},$$

since $\epsilon < 1$.

Suppose now that k is an integer for which (i) is true for every positive ϵ less than 1. Let

$$F(z) = \left(\frac{z}{z-1}\frac{d}{dz}\right)^k S_n(z).$$

Then for all z for which $d_K(z) \geqq \epsilon/2$,

$$|F(z)| < 14\,\frac{2^{k^2+3k}}{(\epsilon/2)^{2k+2}} = 14\,\frac{2^{k^2+5k+2}}{\epsilon^{2k+2}}.$$

Then by the Cauchy inequality, we have, for all z such that $d_K(z) \geqq \epsilon$,

$$|F'(z)| < 14\,\frac{2^{k^2+5k+3}}{\epsilon^{2k+3}}.$$

Since

$$\left|\frac{z}{z-1}\right| \leqq 1 + 1/\epsilon < 2/\epsilon,$$

we have

$$\left|\left(\frac{z}{z-1}\frac{d}{dz}\right)F(z)\right| < 14\,\frac{2^{k^2+5k+4}}{\epsilon^{2k+4}}.$$

Consequently, by mathematical induction, (i) is true for every nonnegative integer k. This completes the proof.

If for each nonnegative integer k we let

$$U_k(z) = \left(\frac{z}{z-1}\frac{d}{dz}\right)^k \frac{z}{1-z},$$

then for the first few values of k we have

$$U_0(z) = \frac{z}{1-z}, \qquad U_1(z) = \frac{z}{(z-1)^3},$$

$$U_2(z) = (-1)\frac{z+2z^2}{(z-1)^5}, \qquad U_3(z) = \frac{z+8z^2+6z^3}{(z-1)^7},$$

$$U_4(z) = (-1)\frac{z+22z^2+58z^3+24z^4}{(z-1)^9},$$

and, in general,

$$U_k(z) = \frac{(-1)^{k+1}Q_k(z)}{(z-1)^{2k+1}},$$

where, for $k>1$, $Q_k(z)$ is a polynomial of degree k with positive integer coefficients.

We are now in a position to state our principle result, the proof of which follows immediately from Lemmas 2 and 3.

THEOREM. *Suppose* $\epsilon>0$. *For every positive integer* k, *the asymptotic formulas*

$$S_n(z) = \sum_{r=0}^{k-1}\left(\frac{1}{n}\right)^r U_r(z) + O(n^{-k}) \tag{3}$$

and

$$T_n(z) = -\sum_{r=0}^{k-1}\left(\frac{1}{n}\right)^r U_r(z) + O(n^{-k}) \tag{4}$$

hold uniformly for $d_K(z)\geqq\epsilon$ *and* $d_J(z)\geqq\epsilon$, *respectively.*

Since every point except $z=1$ belongs either to the complement of J or to the complement of K, we can make use of (i) of Lemma 1 and the characteristic functions $\phi_J(z)$ and $\phi_K(z)$ to combine (3) and (4) and obtain the following:

COROLLARY. *If* $z\neq 1$, *then* $S_n(z)$ *and* $T_n(z)$ *have the asymptotic expansions*

$$S_n(z) \sim n!\left(\frac{e^z}{nz}\right)^n \phi_K(z) + \sum_{r=0}^{\infty}\left(\frac{1}{n}\right)^r U_r(z), \text{ and} \tag{5}$$

$$(6)\qquad T_n(z) \sim n!\left(\frac{e^z}{nz}\right)^n \phi_J(z) - \sum_{r=0}^{\infty}\left(\frac{1}{n}\right)^r U_r(z).$$

If $z=-1$, (5) reduces to Copson's series. Ramanujan's approximation for $S_n(1)$ is, in view of (5), a considerably more singular result than it would otherwise appear.

References

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ON THE LOCAL LINEARIZATION OF DIFFERENTIAL EQUATIONS[1]

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1. Consider the autonomous system of real, nonlinear differential equations

$$(1.1)\qquad x' = Ex + F(x), \quad \text{where} \quad F(|x|) = o(|x|) \text{ as } x \to 0,$$

x is a (Euclidean) vector, $F(x)$ a smooth vector-valued function of x, and E a constant matrix with eigenvalues $e_1, e_2, \cdots$ satisfying

$$(1.2)\qquad \operatorname{Re} e_j \neq 0.$$

Let the solution $\xi(t, x)$ of (1.1) starting at x for $t=0$ be written as

$$(1.3)\qquad T^t\colon x^t = \xi(t, x) = e^{Et}x + X(t, x),$$

where $X(t, x)=o(|x|)$ as $x\to 0$ (for fixed t). Thus if T^t is considered as a map $x\to x^t$, for fixed t, the composition rule

$$(1.4)\qquad T^tT^s = T^{t+s}$$

is valid for small $|x|$. Correspondingly, the linear system

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