ON THE EXTENSION OF MODULAR MAXIMAL IDEALS[1]

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1. **Introduction.** Let B be a complex *commutative* Banach algebra and let A be a subalgebra of B. In [1, §23] and [3] Šilov, about twenty years ago, investigated conditions under which some or all of the modular maximal ideals of A are contained in modular maximal ideals of B. We re-examine this question for *noncommutative* Banach algebras. Here neither A nor B need be commutative and the ideals in question are the modular maximal two-sided ideals[2] of A and B. Simple examples show that, even if A is a commutative subalgebra of B and Šilov's conditions are fulfilled, no m.m. ideal of A need be contained in a m.m. ideal of B. Success can be hoped for only if A is favorably situated in B.

Suppose (for simplicity) that A is closed in B. Consider the set $\mathfrak{Q}$ of all m.m. ideals N of B such that $xy-yx\in N$ for all $x\in A$, $y\in B$. Each $N\in\mathfrak{Q}$ determines a multiplicative linear functional $x\to x(N)$ on A. Let $\mathfrak{Z}$ be the set of m.m. ideals of A which are the null spaces of multiplicative linear functionals on A. The algebra A can be represented homomorphically as an algebra of continuous functions on $\mathfrak{Z}$ in the fashion of Gelfand. Let $\Delta(A)$ be the Šilov boundary of A in $\mathfrak{Z}$. Then[3] each $M\in\Delta(A)$ is contained in a m.m. ideal of B if $\sup|x(N)|=\nu(x)$ for all $x\in A$ where the sup is taken over $\mathfrak{Q}$ and $\nu(x)$ is the spectral radius of x. A sufficient condition for this relation to hold on A is that $xy-yx$ lie in the radical J of B for all $x\in A$, $y\in B$. An example shows that this can take place where A properly contains $C+J$ where C is the center of B.

2. **Algebraic preliminaries.** Let B be a ring with a subring A and denote by $\mathfrak{M}(\mathfrak{N})$ the set of all m.m. ideals of $A(B)$. Let I be a two-sided ideal of B and set $C(I)=\{x\in B\mid xy-yx\in I \text{ for all } y\in B\}$. It is readily seen that $C(I)$ is a subring of B. We assume that $C(I)\supset A$. Given $M\in\mathfrak{M}$ we let $\alpha(M;I)$ denote the set of all finite sums of the

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[2] We shall consider exclusively two-sided ideals and refer to modular maximal two-sided ideals as m.m. ideals.

[3] Consider the special case where B (and so A) is commutative. Here $\mathfrak{Q}$ is the set of all m.m. ideals of B so that automatically $\sup|x(N)|=\nu(x)$ for all x, the sup taken over $\mathfrak{Q}$. As $\mathfrak{Z}$ is here the set of all m.m. ideals of A, our theorem, in this case, reduces to Šilov's theorem [1, p. 213].

form $\sum x_k z_k + u$ where each $x_k \in M$, $z_k \in B$ and $u \in I$. We set $\beta(M; I) = \{w \in B \mid wy \in \alpha(M; I)$ for all $y \in A\}$. Since $C(I) \supset A$ it is clear that $\alpha(M; I)$ can be described as the set of all sums $\sum z_k x_k + u$ so that $\alpha(M; I)$ is a two-sided ideal of B. Also, $\beta(M; I) = \{w \in B \mid yw \in \alpha(M; I)$ for all $y \in A\}$ so that $\beta(M; I)$ is a two-sided ideal of B containing M and I.

We let J denote the radical of B and S its strong radical (the intersection of its m.m. ideals), see [2, p. 59]. Note that $S \supset J$.

2.1. Lemma. *If $\beta(M; I) \neq B$ there exists $N \in \mathfrak{N}$ such that $N \cap A = M$. If I is the strong radical S of B and there exists $N \in \mathfrak{N}$ such that $N \cap A = M$ then $\beta(M; S) \neq B$.*

Suppose that $\beta(M; I) \neq B$ and let j be an identity for A modulo M. Take $z \in B$ and $y \in A$. We can write $zy = yz + v$ where $v \in I$. Then $(jz - z)y = (jy - y)z + jv - v \in \alpha(M; I)$. Therefore $jz - z \in \beta(M; I)$ and likewise $zj - z$ for all $z \in B$. Thus $\beta(M; I)$ is a proper modular two-sided ideal of B containing M and is therefore contained in some $N \in \mathfrak{N}$. Since $j \notin N$ we see that also $N \cap A = M$.

Next take the case $I = S$. Suppose $N \in \mathfrak{N}$ and $N \cap A = M$. Since $N \supset S$ we also see that $N \supset \alpha(M; S)$. If $\beta(M; S) = B$ then $j^2 \in \alpha(M; S) \subset N$. Since $j^2 - j \in M \subset N$ it follows that $j \in N \cap A = M$ which is impossible.

The following example shows that, even for Banach algebras $C(J)$ can be larger than $C + J$ where C is the center of B. Let B be the Banach space of all complex-valued continuous functions on $[0, 1]$ made into a Banach algebra by defining products by the rule $fg(t) = f(0)g(t)$, $0 \leqq t \leqq 1$. For this algebra, $J = S = \{f \in B \mid f(0) = 0\}$, $C = (0)$ and $C(J) = B$.

3. **Extension of maximal ideals.** We adopt the notation of §2 except that A and B are complex Banach algebras (with A algebraically embedded in B). Let $\mathfrak{Q} = \{N \in \mathfrak{N} \mid xy - yx \in N$ for all $x \in A$, $y \in B\}$. For each $N \in \mathfrak{N}$ let $x \to \pi_N(x)$ be the natural homomorphism of B onto B/N. The latter is a simple Banach algebra with an identity. If $N \in \mathfrak{Q}$, the image of A in B/N lies in the center of B/N. But that center is a field and so, by the Gelfand-Mazur theorem, is the set of scalar multiples of its identity $\pi_N(v)$ (see [2, p. 85]). If we write $\pi_N(x) = x(N)\pi_N(v)$ where $x(N)$ is a scalar, the mapping $x \to x(N)$ is a multiplicative linear functional on A (trivial if $N \supset A$). Set

$$\beta(x) = \sup_{N \in \mathfrak{Q}} |x(N)|, \qquad x \in A.$$

Let $\mathfrak{Z}$ denote the subset of $\mathfrak{M}$ consisting of all zero sets of multiplicative linear functionals on A. For each $M \in \mathfrak{Z}$ denote the corresponding functional by $x(M)$. Using the Gelfand theory we can represent A homomorphically as an algebra of continuous functions vanishing at infinity on $\mathfrak{Z}$ where we give to $\mathfrak{Z}$ its weak topology defined by the functions $x(M)$, $x \in A$. We may then speak of the *Šilov boundary* $\Delta(A)$ of A in $\mathfrak{Z}$ in the usual way [**2**, p. 132]. Set

$$\alpha(x) = \sup_{M \in \mathfrak{Z}} |x(M)|, \qquad x \in A.$$

It is clear that

$$(1) \qquad \beta(x) \leqq \alpha(x), \qquad x \in A.$$

For each $x \in A$ let $\|x\|_0(\|x\|)$ be its norm as an element of the Banach algebra $A(B)$. Consider the spectral radii $\nu_A(x) = \lim \|x^n\|_0^{1/n}$ and $\nu_B(x) = \lim \|x^n\|^{1/n}$ of x computed for A and B respectively. The relation $\nu_B(x) \leqq \nu_A(x)$ is automatic. If A is a closed subalgebra of B, $\nu_B(x) = \nu_A(x)$ for $x \in A$. We shall also have occasion to consider the spectrum of x computed in $A(B)$ which we denote by $\mathrm{sp}\,(x|A)$ $(\mathrm{sp}\,(x|B))$.

In the notation above, for $x \in A$ we have, for each positive integer m,

$$|x(N)|\, \|\pi_N(v)\|^{1/m} = \|\pi_N(x^m)\|^{1/m} \leqq \|x^m\|^{1/m}.$$

Letting $m \to \infty$ we see that

$$(2) \qquad \beta(x) \leqq \nu_B(x), \qquad \alpha(x) \leqq \nu_A(x), \qquad x \in A.$$

Let $E(\mathfrak{Q})$ denote the set of $M \in \mathfrak{M}$ for which there exists $N \in \mathfrak{Q}$ such that $N \cap A = M$.

3.1. THEOREM. *The following statements are equivalent.*

(a) $\nu_A(x) = \nu_B(x) = \beta(x)$ *for all* $x \in A$.

(b) $E(\mathfrak{Q}) \supset \Delta(A)$ *and* $\alpha(x) = \nu_A(x)$, $x \in A$.

Suppose (a) holds. It follows from (1) and (2) that $\alpha(x) = \beta(x)$, $x \in A$. Let $k(\mathfrak{Q})$ denote the intersection of the $N \in \mathfrak{Q}$ and let $M_0 \in \Delta(A)$. By Lemma 2.1 it is sufficient to show that $\beta(M_0; k(\mathfrak{Q})) \neq B$ inasmuch as any $N \in \mathfrak{N}$ such that $N \supset \beta(M_0; k(\mathfrak{Q}))$ contains $k(\mathfrak{Q})$ and so lies in $\mathfrak{Q}$. Suppose the contrary. We can then write (where j is an identity for A modulo M_0)

$$(3) \qquad j^2 = \sum_{k=1}^{n} x_k z_k + u,$$

where each $x_k \in M_0$, $z_k \in B$ and $u \in k(\mathfrak{Q})$.

For each $N \in \mathfrak{Q}$ consider the identity $\pi_N(v)$ of B/N. In the quotient algebra norm, $\|\pi_N(v)\| \geqq 1$. An equivalent norm $\|\pi_N(z)\|_1$ may be introduced into B/N so that $\|\pi_N(v)\|_1 = 1$. We suppose that this procedure has been followed for each $N \in \mathfrak{Q}$ and set

$$|||w||| = \sup_{N \in \mathfrak{Q}} \|\pi_N(w)\|_1, \qquad w \in B. \tag{4}$$

Note that $|||w|||$ is defined on all of B whereas $\alpha(x)$ and $\beta(x)$ are only defined on A. It is easy to see that

$$|||x||| = \beta(x), \qquad x \in A. \tag{5}$$

Our argument is now an adaptation of one of Šilov [1, §23]. Without loss of generality we may assume that, in (3), $\beta(x_k) \leqq 1$ for $k = 1, \cdots, n$. Let a be any positive number, $a > \max |||z_k|||$ and $2na > 1$. Note that $j(M_0) = 1$ and $x_k(M_0) = 0$, $k = 1, \cdots, n$. Consider the neighborhood $\mathfrak{B}$ of M_0 in $\mathfrak{Z}$ defined by the inequalities

$$\begin{aligned} &\{M \in \mathfrak{Z} \mid |x_k(M)| < 1/(2na),\ k = 1, \cdots, n\} \\ &\{M \in \mathfrak{Z} \mid |j^2(M) - 1| < 1/3\}. \end{aligned} \tag{6}$$

From [2, p. 135] there exists $y \in A$ such that

$$\sup_{M \in \mathfrak{B}} |y(M)| = 1, \quad \sup_{M \notin \mathfrak{B}} |y(M)| < 1/(2na). \tag{7}$$

Now, for $M \in \mathfrak{B}$, we have $|j^2y(M)| = |j^2(M)|\,|y(M)| \geqq 2|y(M)/3|$ by (6) so that

$$\beta(j^2y) \geqq 2/3. \tag{8}$$

Since $z_ky - yz_k \in k(\mathfrak{Q})$ we can from (3) write

$$j^2y = \sum_{k=1}^{n} (x_ky)z_k + w \tag{9}$$

where $w \in k(\mathfrak{Q})$. Note that $\pi_N(w) = 0$ for each $N \in \mathfrak{Q}$. Therefore we get from (4), (5) and (9) that

$$\beta(j^2y) \leqq \sum_{k=1}^{n} |||x_ky|||\ |||z_k|||. \tag{10}$$

Next, for $M \in \mathfrak{B}$, $|x_ky(M)| \leqq |x_k(M)| < 1/(2na)$ while for $M \notin \mathfrak{B}$, $|x_ky(M)| \leqq \alpha(x_k)|y(M)| \leqq |y(M)| < 1/(2na)$ so that, by (5), $|||x_ky||| \leqq 1/(2na)$. From (10) we see that $\beta(j^2y) \leqq 1/2$. This is contrary to (8) and we have shown that (a) implies (b).

Suppose (b) and let $x \in A$. By the definition of the Šilov boundary there exists $M_0 \in \Delta(A)$ such that $|x(M_0)| = \nu_A(x)$. Let j be an identity for A modulo M_0 and let $N \in \mathfrak{Q}$ have the property that $N \cap A = M_0$. As $j^2 - j \in M_0 \subset N$ and $j(N) \neq 0$ we see that $j(N) = 1$. Suppose $x(M_0) = a$. Then $x - aj \in N$ whence $|x(N)| = \nu_A(x)$. Therefore $\beta(x) \geqq \nu_A(x)$ and, by (2), we see that $\nu_A(x) = \nu_B(x) = \beta(x)$.

3.2. LEMMA. *If $C(J) \supset A$ then $\mathfrak{Q} = \mathfrak{N}$ and $\beta(x) = \nu_B(x)$ for all $x \in A$.*

Since $J \subset S$ it follows from $C(J) \supset A$ that $\mathfrak{Q} = \mathfrak{N}$. Let $x \in A$ and let $a \in \mathrm{sp}(x|B)$, $a \neq 0$. Set $K = \{a^{-1}xy - y \mid y \in B\} + J$. Clearly $K = \{a^{-1}yx - y \mid y \in B\} + J$ and is a modular two-sided ideal of B. Suppose that $K = B$. Then there exists $y \in B$, $z \in J$ such that $a^{-1}x + y - a^{-1}xy = z$. This implies that $a^{-1}x$ is right quasi-regular in B. Likewise $a^{-1}x$ is left quasi-regular and hence quasi-regular in B which is impossible. Thus there exists $N \in \mathfrak{N}$ with $K \subset N$. Since $J \subset N$ it follows that $\pi_N(a^{-1}x)$ is the identity of B/N. Therefore $x(N) = a$. Conversely, if $x(N) = a \neq 0$ for some $N \in \mathfrak{N}$ then $\{a^{-1}xy - y \mid y \in B\}$ lies in N so that $a \in \mathrm{sp}(x|B)$. Thus $\beta(x) = \nu_B(x)$.

Lemma 3.2 gives a sufficient condition for the applicability of Theorem 3.1. It follows immediately that if A is a closed subalgebra of the center C, each $M \in \Delta(A)$ is contained in some $N \in \mathfrak{N}$.

We next consider the strong structure space $\mathfrak{M}(\mathfrak{N})$ of $A(B)$ in its hull-kernel topology [**2**, p. 78]. For the notion of a completely regular Banach algebra see [**2**, p. 83]. We can be more specific than in Theorem 3.1 if A is completely regular.

3.3. THEOREM. *Suppose that A is completely regular and that $\nu_A(x) = \nu_B(x) = \beta(x)$ for all $x \in A$. Then $E(\mathfrak{Q}) = \mathfrak{Z}$.*

Let an identity be adjoined to B forming B_1 and let A_1 denote the corresponding augmentation of A. Let $\mathfrak{M}_1(\mathfrak{N}_1)$ be the strong structure space of $A_1(B_1)$ and let $\mathfrak{Q}_1 = \{N_1 \in \mathfrak{N}_1 \mid wv - vw \in N_1$ for all $w \in A_1$, $v \in B_1\}$. If we write $w = \lambda + x$, $v = \mu + y$ where λ, μ are scalars, $x \in A$ and $y \in B$ we see that $N_1 \in \mathfrak{Q}_1$ if and only if $xy - yx \in N_1$ for all $x \in A$, $y \in B$. The mapping $N_1 \to N_1 \cap B$ takes $\mathfrak{N}_1 \sim \{B\}$ onto $\mathfrak{N}$. We then have $N_1 \in \mathfrak{Q}_1$, $N_1 \neq B$ if and only if $N_1 \cap B \in \mathfrak{Q}$ and all elements of $\mathfrak{Q}$ are obtainable in this way. Next we note that $\mathfrak{N}_1$ is compact [**2**, p. 79] and that $\mathfrak{Q}_1$ is a closed subset of $\mathfrak{N}_1$. The mapping $\sigma: N_1 \to N_1 \cap A_1$ is a mapping of the compact set $\mathfrak{Q}_1$ into $\mathfrak{M}_1$ which is continuous (see [**2**, p. 85]). Also $\mathfrak{M}_1$ is a Hausdorff space [**2**, p. 84] so that $\sigma(\mathfrak{Q}_1)$ is closed in $\mathfrak{M}_1$.

The mapping $\tau: M_1 \to M_1 \cap A$ is a homeomorphism of $\mathfrak{M}_1 \sim \{A\}$ onto $\mathfrak{M}$. But $\sigma(\mathfrak{Q}_1) \sim \{A\}$ is closed in $\mathfrak{M}_1$ so that $\tau[\sigma(\mathfrak{Q}_1) \sim \{A\}]$ is closed in $\mathfrak{M}$. This set is the same as $E(\mathfrak{Q}) = \{N \cap A \mid N \in \mathfrak{Q}, N \not\supset A\}$.

Recall that, by Theorem 3.1, $E(\mathfrak{Q})\supset\Delta(A)$. From our definitions $\Delta(A)$ is dense in $\mathfrak{Z}$ so that also $E(\mathfrak{Q})\supset\mathfrak{Z}$. Since the reverse inequality is clear, the proof is complete.

References

1. I. M. Gel'fand, D. A. Raikov and G. E. Šilov, *Commutative normed rings*, Uspehi Mat. Nauk **1** (1946), 48–146; Amer. Math. Soc. Transl. (2) **5** (1957), 115–220.

2. C. E. Rickart, *General theory of Banach algebras*, Van Nostrand, New York, 1960.

3. G. E. Šilov, *On the extension of maximal ideals*, Dokl. Akad. Nauk SSSR **29** (1940), 83–84.

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SEMI-HOMOGENEOUS FUNCTIONS

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1. **Introduction and statement of results.** A function f is called *homogeneous of degree n with respect to the set A*, or briefly *semi-homogeneous* if

$$f(ax) = a^n f(x) \tag{1.0}$$

is satisfied for all x in the domain of f and all a in A.

With each admissible A there is associated a class of solutions of (1.0). E.g., let R denote the set of all real numbers and let f be a function on R to R. If A consists only of the irrationals, then $f(x) = cx^n$ $(c=f(1))$ is the unique solution of (1.0). On the other hand, if A consists only of the rationals, then in addition to $f(x)=cx^n$, (1.0) has other solutions (e.g., if n is any nonzero integer and $f(x)=x^n$ or 0 accordingly as x is rational or irrational).

We are interested in studying decompositions of the set of admissible A's into classes and in characterizing the solutions of (1.0) corresponding to these classes. In this paper we show how this can be done in a natural way for semi-homogeneous functions of a real variable. We note that in this case our methods apply to

$$f(ax) = p(a)f(x) \qquad (a \in A \subset R), \tag{1.1}$$

where p is a product-preserving function on R to R (cf. [1]). We shall therefore confine our attention to (1.1).

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