# EXAMPLE OF A NONACYCLIC CONTINUUM SEMIGROUP $S$ WITH ZERO AND $S=ESE$

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Throughout this discussion $S$ will denote a compact connected topological semigroup and $E$ will denote the set of idempotents of $S$. The problem to be considered concerns a question posed by Professor A. D. Wallace. In [1], Wallace proves that if $S$ has a left unit, if $I$ is a closed ideal of $S$, and if $L=\square$ or if $L$ is a closed left ideal of $S$, then $H^n(S)\cong H^n(I\cup L)$ for all integers $n$, where $H^n(A)$ denotes the $n$th Alexander-Čech cohomology group of $A$ with coefficients in an arbitrary but fixed group $G$. If $S$ is assumed to have both a left zero and a left unit, then it follows that each closed left ideal $L$, of $S$ is acyclic; that is, $H^p(L)=0$ for all $p\geqq 1$. A dual statement holds for closed right ideals if $S$ has a right unit and right zero. A generalization of the case in which $S$ has a left, right, or two-sided unit, is to require that $S=ES$, $S=SE$, or $S=ESE$, respectively, and Wallace has asked: "If $S$ has a zero, are closed right or left ideals of $S$ necessarily acyclic in the more general situation?" [3]. A negative answer to this question is given here by way of examples, and a theorem is proved giving a necessary and sufficient condition for closed right ideals of $S$ to be acyclic, assuming $S=ESE$ and $S$ has a zero. Following the proof of this theorem is an example of a semigroup not satisfying this condition.

The above-mentioned example shows that even though $S$ is acyclic, it is not necessarily true that all closed right ideals of $S$ are acyclic. Thus the question remains as to whether $S$ is acyclic if $S=ESE$ and $S$ has a zero [3]. Wallace proves in [2] that for such a semigroup $S$, $H^1(S)=0$, however, an example is given here of a semigroup $S$ with zero, $S=ESE$ and $H^2(S)\cong G$ for all groups $G$, showing that this question also has a negative answer. Two further examples are included in this paper which show what can occur if one only assumes that $S=SE$, or $S=ES$. One example is of a semigroup $S$ with zero, $S=SE$ and $H^1(S)\cong G$ for all groups $G$ and the other is an example of a semigroup $S$ with zero and left unit and $S$ contains a closed right ideal $R$ with $H^1(R)\cong G$.

Definition. Let $T$ be a semigroup, $a\in T$ and $R(a)$ the closed right ideal of $T$ generated by $a$. Then $a$ is said to be right codependent on

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$T$ if for any integer $n \geqq 1$, $H^n(T)=0$ implies that $H^n(R(a))=0$.

THEOREM. *Let $S$ be a compact connected semigroup with zero and $S=ESE$. A necessary and sufficient condition that each closed right ideal of $S$ be acyclic is that each $a$ in $S$ be right codependent on $S$.*

The proof of this theorem depends on the following two lemmas. The proofs of these lemmas are paraphrases of the proof of the main theorem in [2] and will be omitted.

LEMMA 1. *Let $S$ be a compact connected semigroup with zero and $S=ES$. Let $n$ be a fixed integer $n \geqq 2$. If $H^{n-1}(R)=0$ for each closed right ideal $R \subset S$, then $H^n(S)=0$.*

LEMMA 2. *Let $S$ be a compact connected semigroup with zero and $S=SE$. Let $n$ be a fixed integer, $n \geqq 1$. If for each $a \in S$, $H^n(aS)=0$ and if for each closed subset $A \subset S$, $H^{n-1}(AS)=0$, then $H^n(R)=0$ for each closed right ideal $R \subset S$. (For $n=1$, reduced groups are to be used.)*

PROOF OF THEOREM. First assume that each $a$ in $S$ is right codependent on $S$. The proof of sufficiency will be by induction on $n$. Let $n=1$. From [2], $H^1(S)=0$, hence it follows that $H^1(aS)=0$ for each $a$ in $S$ since $R(a)=aS$ and each $a$ is right codependent on $S$. Each closed right ideal $R \subset S$ is connected, therefore $H^0(R, r)=0$ for each $r \in R$. Thus, using reduced groups, it follows from Lemma 2 that $H^1(R)=0$ for each closed right ideal $R \subset S$.

Assume now that $H^{k-1}(R)=0$ for each closed right ideal $R$ of $S$ and integer $k \geqq 2$. Then by Lemma 1, $H^k(S)=0$, hence $H^k(aS)=0$ for each $a \in S$. Applying Lemma 2, it follows that $H^k(R)=0$ where $R$ is a closed right ideal of $S$. This completes the proof of sufficiency.

If each closed right ideal of $S$ is acyclic, then $H^n(aS)=0$ for each $a \in S$, integer $n \geqq 1$ and coefficient group $G$ since $aS$ is a closed right ideal. Also $R(a)=aS$ for each $a \in S$ so that it is trivially true that each $a$ in $S$ is right codependent on $S$ which completes the proof of the theorem.

In the following examples, let $I=[0, 1]$ denote the real unit interval and for $x$ and $y$ in $I$ let:

$$x \wedge y = \text{minimum of } x \text{ and } y,$$
$$x \vee y = \text{maximum of } x \text{ and } y,$$
$$xy = \text{real product of } x \text{ and } y.$$

EXAMPLE 1. This is an example of a compact connected semigroup $S$ with zero, $S=ESE$, $S$ is acyclic and there exists an element $p$ in $S$ with $H^1(pS) \cong G$ for all groups $G$. This example shows that there exist semigroups with zero such that each element is not right codependent on $S$. The topological space of $S$ is a two-cell with three closed intervals, $I_1$, $I_2$, and $I_3$, issuing from a common point $z_0$, on the boundary, $B$, of the two-cell. This point $z_0$ is the zero of $S$ and $p\in B\backslash z_0$. By construction $pS=B$ and $B^2=z_0$. In this example, $E=\{e_1, e_2, e_3, z_0\}$ where $e_i$ is the free endpoint of $I_i$.

Example 1 is constructed as follows. Let $\{a, b, c, d, \theta\}$ be a discrete space consisting of five elements. Define spaces $A$, $B$, $C$, $D$ and $S_0$ as follows:

$A=a\times I$, $B=b\times I$, $C=c\times I$, $D=d\times I\times I$, each with the product topology and $S_0=A\cup B\cup C\cup D\cup\{\theta\}$ with the topology on $S_0$ given by the union of the topologies on $A$, $B$, $C$, $D$ and $\{\theta\}$. Define the product $pq$ for $p$ and $q$ in $S_0$ by:

$$pq = \begin{cases} (d, (x\wedge y)r, (x\vee y)), \text{ if } p=(d, x, y)\in D,\ q=(a, r)\in A, \\ (d, (x\vee y), (x\wedge y)r), \text{ if } p=(d, x, y)\in D,\ q=(b, r)\in B, \\ (b, rs), \text{ if } p=(a, r)\in A, \text{ or } p=(b, r)\in B \text{ and } q=(b, s)\in B, \\ (a, rs), \text{ if } p=(b, r)\in B, \text{ or } p=(a, r)\in A \text{ and } q=(a, s)\in A, \\ (d, xr, yr), \text{ if } p=(c, r)\in C \text{ and } q=(d, x, y)\in D, \\ (c, rs), \text{ if } p=(c, r)\in C \text{ and } q=(c, s)\in C, \\ \theta \quad \text{otherwise.} \end{cases}$$

By the definition of the topology on $S_0$, multiplication is continuous and associativity is checked by direct computation. Let $E_0=\{(a, 1), (b, 1), (c, 1), \theta\}$. Then $E_0$ is a set of idempotents in $S_0$ and the claim is made that $S_0=E_0S_0E_0$. This is true since $(a, 1)$ is a two-sided unit for $A$ and a right unit for $(d\times\{(x, y): x\leqq y\})$; $(b, 1)$ is a two-sided unit for $B$ and a right unit for $(d\times\{(x, y): x\geqq y\})$; $(c, 1)$ is a two-sided unit for $C$ and a left unit for $D$; and $\theta^2=\theta$.

Consider now, $I_0=(d\times\{0\}\times I)\cup(d\times I\times\{0\})\cup\{(a, 0), (b, 0), (c, 0), \theta\}$. By direct computation it can be shown that this closed subset of $S_0$ is a two-sided ideal of $S_0$. Let $S=S_0/I_0$ be the Rees quotient of $S_0$ by $I_0$. Then $S$ is a compact connected semigroup with zero and it is clear that $S$ is acyclic. Also the condition $S_0=E_0S_0E_0$ implies that $S=ESE$ where $E$ is the set of idempotents of $S$.

Let $p=(d, 1, 1)$. Then $pS=((d, 1, 1)S_0\cup I_0)/I_0=((d\times I\times\{1\})\cup(d\times\{1\}\times I)\cup I_0)/I_0$ so that $pS$ is homeomorphic to a one-sphere and therefore $H^1(pS)\cong G$ for all groups $G$.

Example 2. This is an example of a compact connected semigroup $S$ with zero, $S=ESE$ and $H^2(S)\cong G$ for all coefficient groups $G$. The topological space of this semigroup is a two-sphere with four closed intervals, $I_i$, $i=1, 2, 3, 4$, issuing from a common point, $z_1$, on the two-sphere. The point $z_1$ is a zero for $S$ and if $e_i$ denotes the free endpoint of $I_i$, then $E=\{e_1, e_2, e_3, e_4, z_1\}$ and multiplication in $S$ has the following properties: (Let $S_1$ denote the two-sphere in $S$; $C_1$ and $C_2$ the two great circles in $S_1$ through $z_1$; $H_1$ and $H_2$ the closed hemispheres determined by $C_1$; and $P_1$, $P_2$ the closed hemispheres determined by $C_2$.) $S_1^2=z_1$; $e_1S=H_1\cup I_1$; $e_2S=H_2\cup I_2$; $Se_3=P_1\cup I_3$; $Se_4=P_2\cup I_4$. Hence $e_1S\cap e_2S=C_1$ is a closed right ideal of $S$ with nontrivial cohomology in dimension one. Similarly, $Se_3\cap Se_4=C_2$ is a closed left ideal of $S$.

$S$ is constructed in the following way: Let $N=\{a, b, c, d, e, \theta\}$ be a discrete space with six elements. Let $N_0=N\setminus\{e, \theta\}$ and let $T=(N_0\times I)\cup(e\times I\times I)\cup\{\theta\}$ with the topology on $T$ given by the union of the topologies of its subsets. For $x$ and $y$ in $[0, 1]$ define $\alpha(x, y)=(x\wedge y\wedge(1-x)\wedge(1-y))$ and define the product $pq$ for $p$ and $q$ in $T$ by:

$$pq = \begin{cases} (e,\ (x\wedge(1-y))-r\alpha(x, y),\ (y\wedge(1-x))-r\alpha(x, y)), & p=(a, r)\in A, \\ & q=(e, x, y)\in S_0,\ =e\times I\times I \\ (e,\ (x\vee(1-y))+r\alpha(x, y),\ (y\vee(1-x))+r\alpha(x, y)), & p=(b, r)\in B, \\ & q=(e, x, y)\in S_0 \\ (a,\ r+s-rs), & p=(a, r)\in A \text{ and } q=(a, s)\in A \text{ or } q=(b, s)\in B, \\ (b,\ r+s-rs), & p=(b, r)\in B \text{ and } q=(a, s)\in A \text{ or } q=(b, s)\in B, \\ (e,\ 1-s+(x\vee y)s,\ (x\wedge y)s), & p=(e, x, y)\in S_0,\ q=(c, s)\in C, \\ (e,\ (x\wedge y)s,\ 1-s+(x\vee y)s), & p=(e, x, y)\in S_0,\ q=(d, s)\in D, \\ (d,\ rs), & p=(c, r)\in C \text{ or } p=(d, r)\in D \text{ and } q=(d, s)\in D, \\ (c,\ rs), & p=(d, r)\in D \text{ or } p=(c, r)\in C \text{ and } q=(c, s)\in C, \\ \theta & \text{otherwise.} \end{cases}$$

By the definition of the topology on $T$, it is clear that multiplication is continuous since it involves continuous operations of real numbers. By direct computation it is seen that multiplication is also associative and therefore $T$ is a compact semigroup. Let $S_i$, $i=1, 2, 3, 4$, subsets of $e\times I\times I$ be defined by:

$$S_1 = \{(e, x, y): 0 \leqq x \leqq y \leqq x + y \leqq 1\},$$
$$S_2 = \{(e, x, y): 0 \leqq x \leqq y \leqq 1 \leqq x + y\},$$
$$S_3 = \{(e, x, y): 0 \leqq y \leqq x \leqq 1 \leqq x + y\},$$
$$S_4 = \{(e, x, y): 0 \leqq y \leqq x \leqq x + y \leqq 1\},$$

and let $E_0=\{(a, 0), (b, 0), (c, 1), (d, 1), \theta\}$. $E_0$ is a set of idempotents in $T$ and the claim is made that $T=E_0TE_0$. This follows from the following equalities:

$a\times I=(a,0)(a\times I)(a,0)$; $b\times I=(b,0)(b\times I)(b,0)$; $c\times I=(c,1)(c\times I)(c,1)$; $d\times I=(d,1)(d\times I)(d,1)$; $\theta^2=\theta$ and $e\times I\times I=\bigcup\{S_i: i=1,2,3,4\}=(a,0)S_1(d,1)\cup(b,0)S_2(d,1)\cup(b,0)S_3(c,1)\cup(a,0)S_4(c,1)$. This proves that $T=E_0TE_0$ as claimed.

Now let $I_0=\{(a,1),(b,1),(c,0),(d,0),\theta\}\cup(e\times F(I\times I))$ where $F(I\times I)$ denotes the boundary of $I\times I$ in the Euclidean plane. It can be shown that this closed subset of $T$ is a two-sided ideal of $T$, hence $S=T/I_0$ is a compact connected semigroup as described above. Also $S=ESE$, since $T=E_0TE_0$ and $S$ has a zero.

EXAMPLE 3. This example is of a semigroup $S=SE$ which is compact connected, has a zero and $H^1(S)\cong G$ for all groups $G$. $S$ is a subsemigroup of the semigroup in Example 1 and the topological space of $S$ is a circle with two closed intervals issuing from a common point of the circle.

In the terminology of Example 1, consider the following closed subsemigroup, $T$, of $S_0$:

$T=A\cup B\cup(d\times F(I\times I))\cup\{\theta\}$. Then $T=T(a,1)\cup T(b,1)\cup\{\theta\}$ so that $T=TE_1$ where $E_1$ is the set of idempotents in $T$. Let $I_1=\{(a,0),(b,0),\theta\}\cup(d\times\{0\}\times I)\cup(d\times I\times\{0\})$. Then $I_1=T\cap I_0$ is a closed ideal of $T$ and $S=T/I_1$ is a compact connected semigroup with zero and $S=SE$. Clearly $S$ is topologically as described above, so that $H^1(S)\cong G$.

EXAMPLE 4. This final example is of a semigroup $S$ with zero and left unit and $S$ contains a closed right ideal $R$ such that $H^1(R)\cong G$, for all groups $G$.

Let $S=(\{0\}\times I\times I)\cup(I\times\{0\}\times I)$ and define multiplication in $S$ by $(x,y,z)(r,s,t)=(xr,xs,zt)$. $S$ can be represented by the following matrix semigroup:

$$\left\{\begin{pmatrix} x & y & 0\\ 0 & 0 & 0\\ 0 & 0 & z\end{pmatrix} : (x,y,z)\in S\right\}$$

so that multiplication in $S$ is continuous and associative. Clearly $(0,0,0)$ is a zero for $S$ and $(1,0,1)$ is a left unit for $S$. By the definition of multiplication it follows that any subset of $(\{0\}\times I\times I)$ containing $(\{0\}\times\{0\}\times I)$ is a right ideal of $S$ and, in particular, $R=$ the boundary of $(\{0\}\times I\times I)$ is a closed right ideal of $S$ and $H^1(R)\cong G$ for all coefficient groups $G$.

In these four examples it might be noted that the set of idempotents in each semigroup was a finite discrete set. It might be of interest to know if there exists a semigroup $S=ESE$ which is compact connected, has a zero, is not acyclic and such that the set of idempotents is connected.

## Bibliography


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# CONFORMAL VECTOR FIELDS IN COMPACT RIEMANNIAN MANIFOLDS

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1. **Introduction.** Let $V^n$ be a compact Riemannian manifold of dimension $n$ and of class $C^3$. Let $g_{ij}(x)$ of class $C^2$ be the coefficients of the fundamental metric which is assumed to be positive definite. Let $\Gamma^h_{ij}$ be the Christoffell symbol, $R_{ijhk}$ the curvature tensor and $R_{ij}$ the Ricci tensor.

Let $\phi$ be an arbitrary scalar invariant, $\xi^i$ an arbitrary vector field and $\xi_{i_1i_2\cdots i_p}$ an arbitrary anti-symmetric tensor field of order $p$, all of class $C^2$ in $V^n$. We shall make use of the following results obtained by S. Bochner and K. Yano [1, pp. 31, 51, 69]:

(1.1) $(\Delta\phi\geqq 0$ everywhere in $V^n)\Rightarrow(\phi=$ constant everywhere in $V^n)$.

$$\int_{V^n}\xi^i{}_{,i}\,dv = 0. \tag{1.2}$$

$$\int_{V^n}(R_{ij}\xi^i\xi^j + \xi^i{}_{,j}\,\xi^j{}_{,i} - \xi^i{}_{,i}\,\xi^j{}_{,j})dv = 0. \tag{1.3}$$

$$\int_{V^n}(F\{\xi_{i_1i_2\cdots i_p}\} + \xi^{ii_2\cdots i_p,j}\,\xi_{ji_2\cdots i_p,i} - \xi^{ii_2\cdots i_p}{}_{,i}\,\xi^j{}_{i_2\cdots i_p,j})dv=0 \tag{1.4}$$

where

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