# A NOTE ON THE ENTROPY OF SKEW PRODUCT TRANSFORMATIONS[1]

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Let $(X, \mathfrak{X}, \lambda)$ and $(Y, \mathfrak{Y}, \mu)$ be two Lebesgue spaces with $\mathfrak{X}$ and $\mathfrak{Y}$ the fields of measurable subsets of $X$ and $Y$ respectively. $\lambda$ and $\mu$ are countably additive measures on $\mathfrak{X}$ and $\mathfrak{Y}$ respectively with $\lambda(X) = \mu(Y) = 1$. Let $(Z, \mathfrak{Z}, \nu) = (X \times Y, \mathfrak{X} \times \mathfrak{Y}, \lambda \times \mu)$ denote the direct product of the above measure spaces. Let $\phi$ be a measure preserving transformation on $X$, and for each $x \in X$ let $\psi_x$ be measure preserving transformation on $Y$. If the family $\{\psi_x : x \in X\}$ of measure preserving transformations satisfies certain measurability conditions (see [**2**, pp. 83, 84]), then it can be shown that the transformation $T$ defined by

$$T(x, y) = (\phi x, \psi_x y)$$

is a measure preserving transformation on $Z$. $T$ is called the skew product transformation of $\phi$ with the family $\{\psi_x : x \in X\}$.

The purpose of this work is to compute the entropy $h(T)$. (For definition of entropy of a measure preserving transformation and the associated notation consult [**3**] and [**4**].) The natural conjecture is

$$h(T) = h(\phi) + \int_X h(\psi_x)\lambda(dx). \tag{*}$$

This conjecture is substantiated in several instances. When $\psi_x = \psi$ for all $x \in X$, (*) reduces to the formula for direct product transformations (see [**4**] formula ($\beta$)); i.e.,

$$h(T) = h(\phi) + h(\psi).$$

For $\phi = I$ the identity transformation on $X$, (*) reduces to the case of decomposition of a measure preserving transformation into components (see [**4**] formula ($\epsilon$)); i.e.,

$$h(T) = \int_X h(\psi_x)\lambda(dx).$$

When $Y$ = unit interval and $\psi_x y = y + \alpha(x)$ (mod 1) where $\alpha(\cdot)$ is some real-valued measurable function on $X$, Abramov [**1**] has shown

$$h(T) = h(\phi)$$

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[1] The material in this paper was included in the author's doctoral dissertation submitted to Yale University (1961) under the direction of Professor S. Kakutani.

which is again a special case of (*) since $h(\psi_x)=0$, $x\in X$.

In general (*) is not true. However, we shall derive a formula which differs from (*) in the function occurring within the integral.

Let $\mathfrak{X}_k$, $k=1, 2, \cdots$ be an increasing sequence of finite subfields of $\mathfrak{X}$ whose union generates $\mathfrak{X}$ and let $\mathfrak{Z}_m$, $m=1, 2, \cdots$ be an increasing sequence of finite subfields of $\mathfrak{Z}$ whose union generates $\mathfrak{Z}$. Let ${}^n_T\mathfrak{Z}_m$ denote $\mathfrak{Z}_m \vee T\mathfrak{Z}_m \vee \cdots \vee T^{n-1}\mathfrak{Z}_m$. Denote by $(\mathfrak{Z}_m)_x$ the field of subsets of $Y$ which consists of $x$-sections of sets in $\mathfrak{Z}_m$. We observe

$$
\begin{aligned}
({}^n_T\mathfrak{Z}_m)_x = (\mathfrak{Z}_m)_x \vee \psi_{\phi^{-1}x}(\mathfrak{Z}_m)_{\phi^{-1}x} \vee \cdots \\
\vee\, \psi_{\phi^{-1}x}\psi_{\phi^{-2}x} \cdots \psi_{\phi^{-n+1}x}(\mathfrak{Z}_m)_{\phi^{-n+1}x}.
\end{aligned} \tag{1}
$$

We shall employ an ambiguity whose meaning will be clear in context by having the symbols $\mathfrak{X}_k$ represent either fields of measurable subsets of $X$ or fields of cylinder sets in $\mathfrak{Z}$ based on subsets of $X$ in $\mathfrak{X}_k$. Keeping this ambiguity in mind consider the following relation between mean entropy and mean (conditional) entropy of finite fields (see [**4**, p. 980])

$$
\begin{aligned}
H({}^n_T\mathfrak{X}_k \vee {}^n_T\mathfrak{Z}_m) &= H({}^n_T\mathfrak{X}_k) + H({}^n_T\mathfrak{Z}_m \,|\, {}^n_T\mathfrak{X}_k) \\
&= H({}^n_\phi\mathfrak{X}_k) + H({}^n_T\mathfrak{Z}_m \,|\, {}^n_T\mathfrak{X}_k).
\end{aligned}
$$

Dividing by $n$ and observing ${}^n_T\mathfrak{X}_k \subseteq \mathfrak{X}$ we have the inequality

$$
\frac{H({}^n_T\mathfrak{X}_k \vee {}^n_T\mathfrak{Z}_m)}{n} \geqq \frac{H({}^n_\phi\mathfrak{X}_k)}{n} + \frac{H({}^n_T\mathfrak{Z}_m \,|\, \mathfrak{X})}{n}. \tag{2}
$$

The definition of mean (conditional) entropy yields

$$
H({}^n_T\mathfrak{Z}_m \,|\, \mathfrak{X}) = \int_X H(({}^n_T\mathfrak{Z}_m)_x)\lambda(dx).
$$

By replacing the function in the integral with (1) and substituting in (2) we get

$$
\begin{aligned}
\frac{H({}^n_T\mathfrak{X}_k \vee {}^n_T\mathfrak{Z}_m)}{n} &\geqq \frac{H({}^n_\phi\mathfrak{X}_k)}{n} \\
&+ \int_X \frac{H((\mathfrak{Z}_m)_x \vee \psi_{\phi^{-1}x}(\mathfrak{Z}_m)_{\phi^{1-x}} \vee \cdots \vee \psi_{\phi^{-1}x}\psi_{\phi^{-2}x} \cdots \psi_{\phi^{-n+1}x}(\mathfrak{Z}_m)_{\phi^{-n+1}x})}{n} \\
&\cdot \lambda(dx).
\end{aligned} \tag{3}
$$

The following identity can be established (see for instance [**3**, p. 33]):

$$
\begin{aligned}
&H((\mathcal{Z}_m)_x \vee \psi_{\phi^{-1}x}(\mathcal{Z}_m)_{\phi^{-1}x} \vee \cdots \vee \psi_{\phi^{-1}x}\psi_{\phi^{-2}x} \cdots \psi_{\phi^{-n+1}x}(\mathcal{Z}_m)_{\phi^{-n+1}x}) \\
(4) \quad &= \sum_{j=0}^{n-1} H((\mathcal{Z}_m)_{\phi^{-j}x} \mid \psi_{\phi^{-j-1}x}(\mathcal{Z}_m)_{\phi^{-j-1}x} \vee \cdots \\
&\qquad\qquad \vee \psi_{\phi^{-j-1}x} \cdots \psi_{\phi^{-n+1}x}(\mathcal{Z}_m)_{\phi^{-n+1}x}).
\end{aligned}
$$

As $n$ tends to $\infty$,

$$H((\mathcal{Z}_m)_{\phi^{-j}x} \mid \psi_{\phi^{-j-1}x}(\mathcal{Z}_m)_{\phi^{-j-1}x} \vee \cdots \vee \psi_{\phi^{-j-1}x} \cdots \psi_{\phi^{-n+1}x}(\mathcal{Z}_m)_{\phi^{-n+1}x})$$

decreases to a limit which we denote by $f_\phi(x, \mathcal{Z}_m, j)$. Likewise, as $n \to \infty$,

$$
\begin{aligned}
&\int_X H((\mathcal{Z}_m)_{\phi^{-j}x} \mid \psi_{\phi^{-j-1}x}(\mathcal{Z}_m)_{\phi^{-j-1}x} \vee \cdots \\
&\qquad\qquad \vee \psi_{\phi^{-j-1}x} \cdots \psi_{\phi^{-n+1}x}(\mathcal{Z}_m)_{\phi^{-n+1}x})\lambda(dx)
\end{aligned}
$$

tends to

$$\int_X f_\phi(x, \mathcal{Z}_m, j)\lambda(dx).$$

By virtue of the fact that $\phi$ is measure preserving it follows that

$$(5) \qquad \int_X f_\phi(x, \mathcal{Z}_m, j)\lambda(dx) = \int_X f_\phi(x, \mathcal{Z}_m, 0)\lambda(dx).$$

Having outgrown the need for a function of three variables we replace $f_\phi(x, \mathcal{Z}_m, 0)$ by simply $f_\phi(x, \mathcal{Z}_m)$. From (4) and (5) and the fact that ordinary convergence implies Cesaro convergence we obtain

$$(6) \qquad \int_X \frac{H((\overset{n}{{}_T}\mathcal{Z}_m)_x)}{n}\lambda(dx) \to \int_X f_\phi(x, \mathcal{Z}_m)\lambda(dx)$$

as $n \to \infty$. Taking limits in (3) as $n \to \infty$ yields

$$(7) \qquad h(T) \geqq h(T, \mathfrak{X}_k \vee \mathcal{Z}_m) \geqq h(\phi, \mathfrak{X}_k) + \int_X f_\phi(x, \mathcal{Z}_m)\lambda(dx).$$

Then letting $k \to \infty$ we get

$$(8) \qquad h(T) \geqq h(\phi) + \int_X f(x, \mathcal{Z}_m)\lambda(dx).$$

In order to obtain the reverse inequality consider

$$
(9)\qquad
\begin{aligned}
\frac{H({}_{T}^{nl}\mathcal{Z}_m)}{nl} &\leqq \frac{H({}_{T}^{nl}\mathfrak{X}_k \vee {}_{T}^{nl}\mathcal{Z}_m)}{nl} = \frac{H({}_{T}^{nl}\mathfrak{X}_k)}{nl} + \frac{H({}_{T}^{nl}\mathcal{Z}_m \mid {}_{T}^{nl}\mathfrak{X}_k)}{nl} \\
&\leqq \frac{H({}_{\phi}^{nl}\mathfrak{X}_x)}{nl} + \frac{H({}_{T}^{nl}\mathcal{Z}_m \mid {}_{T}^{nl}\mathfrak{X}_k)}{nl}.
\end{aligned}
$$

Now

$$
(10)\qquad
\begin{aligned}
H({}_{T}^{nl}\mathcal{Z}_m \mid {}_{T}^{nl}\mathfrak{X}_k) &\leqq \sum_{i=0}^{n-1} H(T^{il}({}_{T}^{l}\mathcal{Z}_m) \mid {}_{T}^{nl}\mathfrak{X}_k) \leqq \sum_{i=0}^{n-1} H(T^{il}({}_{T}^{l}\mathcal{Z}_m) \mid T^{il}\mathfrak{X}_k) \\
&\leqq nH({}_{T}^{l}\mathcal{Z}_m \mid \mathfrak{X}_k).
\end{aligned}
$$

Combining (9) and (10) we have

$$
(11)\qquad \frac{(H{}_{T}^{nl}\mathcal{Z}_m)}{nl} \leqq \frac{H({}_{\phi}^{nl}\mathfrak{X}_k)}{nl} + \frac{H({}_{T}^{l}\mathcal{Z}_m \mid \mathfrak{X}_k)}{l}.
$$

Letting $n \to \infty$ in (11)

$$
(12)\qquad h(T, \mathcal{Z}_m) \leqq h(\phi, \mathfrak{X}_k) + \frac{H({}_{T}^{l}\mathcal{Z}_m \mid \mathfrak{X}_k)}{l},
$$

and letting $k \to \infty$ in (12)

$$
h(T, \mathcal{Z}_m) \leqq h(\phi) + \frac{H({}_{T}^{l}\mathcal{Z}_m \mid \mathfrak{X})}{l},
$$

or equivalently,

$$
(13)\qquad h(T, \mathcal{Z}_m) \leqq h(\phi) + \int_X \frac{H(({}_{T}^{l}\mathcal{Z}_m)_x)}{l}\,\lambda(dx).
$$

Next letting $l \to \infty$ and combining the result with (8) we have

$$
(14)\qquad h(T, \mathcal{Z}_m) \leqq h(\phi) + \int_X f_\phi(x, \mathcal{Z}_m)\lambda(dx) \leqq h(T).
$$

Now

$$
f_\phi(x, \mathcal{Z}_m) = \lim_{n \to \infty} \frac{H(({}_{T}^{n}\mathcal{Z}_m)_x)}{n},
$$

and it is clear that $f(x, \mathcal{Z}_m)$ increases with $m$ to a possibly infinite but well defined limit $f_\phi(x)$. Since $\lim_{m\to\infty} h(T, \mathcal{Z}_m) = h(T)$, it follows from (14) that

$$h(T) = h(\phi) + \int_X f_\phi(x)\lambda(dx). \tag{**}$$

Of course we would like to establish

$$\int_X f_\phi(x)\lambda(dx) = \int_X h(\psi_x)\lambda(dx) \tag{***}$$

where

$$f_\phi(x) = \lim_{m\to\infty} f_\phi(x, \mathcal{Z}_m)$$

$$f_\phi(x, \mathcal{Z}_m) = \lim_{n\to\infty} \frac{H((\mathcal{Z}_m)_x \vee \psi_{\phi^{-1}x}(\mathcal{Z}_m)_{\phi^{-1}x} \vee \cdots \vee \psi_{\phi^{-1}x}\psi_{\phi^{-2}x} \cdots \psi_{\phi^{-n+1}x}(\mathcal{Z}_m)_{\phi^{-n+1}x})}{n}$$

$$h(\psi_x) = \lim_{m\to\infty} h(\psi_x, \mathcal{Z}_m)$$

$$h(\psi_x, \mathcal{Z}_m) = \lim_{n\to\infty} \frac{H((\mathcal{Z}_m)_x \vee \psi_x(\mathcal{Z}_m)_x \vee \cdots \vee \psi_x^{n-1}(\mathcal{Z}_m)_x)}{n}.$$

The quantities $h(\psi_x, \mathcal{Z}_m)$ and $f(x, \mathcal{Z}_m)$ are different by the nature of their definitions. Perhaps only mild restrictions are required so that the differences can be eliminated by integration to yield (***). The following example, however, reveals that in general (***) is not true: let $X = X_1 \cup X_2$ where $m(X_1) = m(X_2) = \frac{1}{2}$; let $\psi_x = \psi$, $x \in X_1$ and $\psi_x = \psi^{-1}$, $x \in X_2$ where $\psi$ is a measure preserving transformation on $Y$ such that $h(\psi) \neq 0$; and let $\phi$ be a measure preserving transformation on $X$ such that $\phi X_1 = X_2$, $\phi X_2 = X_1$ and $\phi^2 = I$. Then for $T: (x, y) \to (\phi x, \psi_x y)$ we have $T^2$ is the identity transformation on $X \times Y$ so that $h(T) = 0$; but $h(\phi) + \int_X h(\psi_x)\lambda(dx) = h(\psi) \neq 0$.

## References

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