A CONVERGENCE PROBLEM FOR CONTINUED FRACTIONS[1]

T. L. HAYDEN

In this paper a sequence V of regions in the complex plane is called an *admissible sequence* provided that:

(i) for $n \geqq 2$, V_n is either a circle with center the origin plus its interior (C_0+int.), or a circle with center the origin plus its exterior (C_0+ext.), and

(ii) the continued fraction

$$\frac{1/1}{} + \frac{a_2/1}{} + \frac{a_3/1}{} + \frac{a_4/1}{} + \cdots \tag{1.1}$$

converges if for $n \geqq 2$, $a_n \in V_n$.

The problem that is raised in this paper, and to which the following theorems contribute a partial solution, is the problem of finding all admissible sequences for (1.1). The collection of all admissible sequences is denoted by AS.

Before stating the theorems, it is convenient to have some additional notation and definitions. The continued fraction (1.1) is considered as being generated by the sequence t of linear fractional transformations defined by:

$$t_1(z) = 1/z, \qquad t_n(z) = 1 + a_n/z \qquad n \geqq 2. \tag{1.2}$$

The sequence T of linear fractional transformations is defined by:

$$T_1(z) = t_1(z), \qquad T_n(z) = T_{n-1}(t_n(z)) \qquad n \geqq 2, \tag{1.3}$$

so that

$$T_n(z) = \frac{A_{n-1}z + a_n A_{n-2}}{B_{n-1}z + a_n B_{n-2}} \tag{1.4}$$

where $a_1 = 1$ and

$$\begin{aligned} A_0 &= 0, \ A_1 = 1; \qquad A_n = A_{n-1} + a_n A_{n-2}; \\ B_0 &= 1, \ B_1 = 1; \qquad B_n = B_{n-1} + a_n B_{n-2}. \end{aligned} \tag{1.5}$$

The nth approximant f_n of (1.1) is $T_n(1) = A_n/B_n$ for $n \geqq 1$.

THEOREM 1. *Suppose V is a sequence such that for each integer $n > 1$*

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either V_n *is a* C_0+int. *or* V_n *is a* C_0+ext., *and there exists an integer* $p>1$ *such that* V_p *and* V_{p+1} *are not bounded; then there exists a sequence* a *of complex numbers such that* $a_n \in V_n$ *for* $n>1$ *and the continued fraction* (1.1) *diverges.*

The proof of Theorem 1 is simplified by the following lemma.

LEMMA 1. *If each of* R *and* S *is a positive number,* u *is a complex number not* 1, *and* v *is a complex number not* 0, *then there exists a complex number* a *and a complex number* b *such that* $|a|>R$, $|b|>S$ *and* $u=1+a/(1+b/v)$.

PROOF. Let n be a positive number such that $n|u-1|>R$ and $(n-1)|v|>S$. Then, if $a=n(u-1)$ and $b=(n-1)v$ the conclusion of Lemma 1 follows.

PROOF OF THEOREM 1. Suppose p is an integer such that V_p and V_{p+1} are not bounded. If the continued fraction

$$(1.7) \qquad 1/1 \underset{+}{} a_{p+2}/1 \underset{+}{} a_{p+3}/1 \underset{+}{} \cdots$$

diverges in the sense that the approximants have more than one limit point, then (1.1) diverges. Therefore, we assume that (1.7) does not diverge in the above mentioned sense and let a_{p+2} be a complex number in V_{p+2} so that (1.7) converges to a complex number v not 0. If $p=2$ there exists, by Lemma 1, an a_2 in V_2 and an a_3 in V_3 such that $0=1+a_2/(1+a_3/v)$. If $p>2$, let $a_2, a_3, a_4, \cdots, a_{p-1}$ be nonzero complex numbers in $V_2, V_3, V_4, \cdots, V_{p-1}$, respectively, such that B_{p-2} and B_{p-3} are not 0 and $-a_{p-1}B_{p-3}/B_{p-2}$ is not 1. By Lemma 1, there exists an a_p in V_p and an a_{p+1} in V_{p+1} such that, $-a_{p-1}B_{p-3}/B_{p-2} = 1+a_p/(1+a_{p+1}/v)$. We now note that by (1.3) and (1.4) $T_1(0)=\infty$ and $T_{p-1}(-a_{p-1}B_{p-3}/B_{p-2})=\infty$ for $p>2$. Hence, the approximants $f_n \to \infty$ as $n\to\infty$.

REMARK 1. Lane and Wall [3] completely settled the problem of finding all admissible sequences where each region of the sequence is bounded by showing that, for $V\in$AS (and V_n bounded), it is necessary and sufficient that there exist a sequence g of positive numbers less than 1 such that V_n: $|z| \leqq (1-g_{n-1})g_n$.

THEOREM 2. *Suppose* V *is a sequence such that for* $n \geqq 2$

(i) *Either* V_n *is a* C_0+int. *or* V_n *is a* C_0+ext.,

(ii) *At least one of* V_n *or* V_{n+1} *is a* C_0+int., *and*

(iii) *There exists a number* g_{n-1} *and a number* r_n *such that,*

(a) $0<g_{n-1}<1$, $0<r_n\leqq 1$,

$$V_n\colon \begin{cases} |z| \leqq r_n(1-g_{n-1})g_n & \text{if } V_n \text{ is bounded,} \\ |z| \geqq (1+g_{n-1})(2-g_n) & \text{if } V_n \text{ is not bounded, and} \end{cases}$$

(b) *if p is an integer such that V_{p+1} is not bounded, and M is the collection of all such integers, then either M is finite or $\prod_{k \text{ in } M} r_k = 0$. Then $V \in \mathrm{AS}$.*

LEMMA 2.1. *Suppose that V is a sequence which satisfies the hypothesis of Theorem 2, $a_n \in V_n$, and for $n \geqq 1$,*

$$Z_n: \begin{cases} |z-1| \leqq 1 - g_n & \text{if } V_{n+1} \text{ is bounded,} \\ |z| \geqq g_n & \text{if } V_{n+1} \text{ is not bounded,} \end{cases}$$

then $t_n(Z_n) \subset Z_{n-1}$ for $n \geqq 2$.

PROOF. For convenience the proof is divided into two cases: V_n bounded and V_n not bounded.

Suppose V_n is bounded; consequently Z_{n-1}: $|z-1| \leqq 1-g_{n-1}$. If $z \in Z_n$ the minimum value of $|z|$ (for V_{n+1} bounded or not bounded) is g_n. It follows that for z in Z_n and a_n in $V_n |t_n(z)-1| = |a_n|/|z| \leqq r_n(1-g_{n-1})g_n/g_n \leqq 1-g_{n-1}$. Hence, $t_n(Z_n) \subset Z_{n-1}$ in case V_n is bounded.

In case V_n is not bounded, so that V_{n+1} is bounded, then Z_{n-1}: $|z| \geqq g_{n-1}$ and Z_n: $|z-1| \leqq 1-g_n$. The maximum value of $|z|$ for z in Z_n is $2-g_n$. Let $z \in Z_n$ and $a_n \in V_n$, then $|t_n(z)-1| \geqq (1+g_{n-1})(2-g_n)/(2-g_n)$; so that $|t_n(z)| \geqq g_{n-1}$, and hence the lemma is true for V_n not bounded.

The following lemma constructs a sequence U of circular regions each containing the next such that the nth approximant f_n of (1.1) is in U_n.

LEMMA 2.2. *Suppose V is a sequence which satisfies the hypothesis of Theorem 2, $a_n \in V_n$, Z is a sequence which satisfies the hypothesis of Lemma 2.1, and U is the sequence defined by: $U_n = T_n(Z_n)$ for $n \geqq 1$. Then, for $n \geqq 1$, $U_{n+1} \subset U_n$. Furthermore, $f_n \in U_n$.*

PROOF. We need only note, from Lemma 2.1, that $t_n(Z_n) \subset Z_{n-1}$ and, by (1.3), $T_n(z) = T_{n-1}(t_n(z))$; consequently, since $U_n = T_n(Z_n) = T_{n-1}(t_n(Z_n))$ and $U_{n-1} = T_{n-1}(Z_{n-1})$, therefore $U_n \subset U_{n-1}$ for $n \geqq 2$. From the definition of Z_n it is clear that $1 \in Z_n$, and since $T_n(1) = f_n$ it follows that $f_n \in U_n$.

LEMMA 2.3. *Suppose V is a sequence which satisfies the hypothesis of Theorem 2 and $a_n \in V_n$. Then, for $p \geqq 2$, $|B_p/B_{p-1} - 1| \leqq g_p$, if V_p is bounded, and $|B_p/B_{p-1} - 1| \geqq 2 - g_p$, if V_p is not bounded.*

PROOF. By (1.5), $B_0 = 1$, $B_1 = 1$, and $B_p = B_{p-1} + a_p B_{p-2}$, so that the lemma is true for $p = 2$. Suppose the lemma is not true for some set M of positive integers. Let $k+1$ be the least integer in M. Now by the assumption that $k+1$ is the least integer for which the inequalities fail, we see that B_k and B_{k-1} are not 0, and hence by (1.5) $|B_{k+1}/B_k - 1| = |a_{k+1}|/|B_k/B_{k-1}|$.

We shall now consider two cases: V_{k+1} bounded and V_{k+1} not bounded. In case V_{k+1} is bounded $|a_{k+1}| \leqq (1-g_k)g_{k+1}$. Also, $|B_k/B_{k-1}| \geqq 1-g_k$, for V_k bounded or unbounded. Hence $|B_{k+1}/B_k - 1| \leqq g_{k+1}$.

In case V_{k+1} is not bounded, V_k is bounded, so that $|a_{k+1}| \geqq (1+g_k)(2-g_{k+1})$ and $|B_k/B_{k-1}| \leqq 1+g_k$. Hence $|B_{k+1}/B_k - 1| \geqq 2-g_{k+1}$ and the statement in the lemma is true for $k+1$. Consequently, the set M does not exist.

We note that for $p \geqq 1$, $B_p \neq 0$ and $|B_p| \geqq |B_{p-1}|(1-g_p)$.

PROOF OF THEOREM 2. In case V_{n+1} is bounded for $n \geqq 1$ or M is finite, then the proof follows from Remark 1 and Lemma 2.2. Suppose $p \in M$, $p > 2$, and M is infinite. Let R_p and C_p denote the radius and center respectively of U_p. Since g_p is on the boundary of Z_p and $T_p(Z_p) = U_p$, hence $R_p = |C_p - T_p(g_p)|$. Since points inverse in the circle Z_p are mapped by $T_p(z)$ into points inverse in the circle U_p it follows that $C_p = T_p(g_p^2[B_{p-1}]^*/[-a_pB_{p-2}]^*)$, inasmuch as $T_p(-a_pB_{p-2}/B_{p-1}) = \infty$. We may then use these expressions and (1.4) to show that

$$R_p = \frac{|a_p g_p| \cdot |A_{p-1}B_{p-2} - A_{p-2}B_{p-1}|}{|g_p^2 \cdot |B_{p-1}|^2 - |a_p|^2 \cdot |B_{p-2}|^2|}.$$

A similar argument, plus the use of (1.5), shows that

$$R_{p-1} = \frac{|1 - g_{p-1}| \cdot |A_{p-1}B_{p-2} - A_{p-2}B_{p-1}|}{||B_{p-1}|^2 - (1-g_{p-1})^2 \cdot |B_{p-2}|^2|}.$$

Since $|a_p| \leqq r_p(1-g_{p-1})g_p$, Lemma 2.3 is sufficient to show that $R_p/R_{p-1} \leqq r_p$. By Lemma 2.2, $U_{p+1} \subset U_p$ and hence the sequence $R_1, R_2, R_3, \cdots$ is a nonincreasing sequence, consequently, the condition $\prod_{k \text{ in } M} r_k = 0$ is sufficient to show that the regions of the sequence U have only one point in common, and hence (1.1) converges.

THEOREM 3. *Suppose V is a sequence such that conditions* (i), (ii), *and* (iii)-(a) *of the hypothesis of Theorem* 2 *are satisfied with $r_n = 1$ for $n \geqq 1$, and $\sum_{r=1}^{r=\infty} m_1, m_2, m_3, \cdots, m_r$ converges where*

$$m_n = \begin{cases} g_n/(1-g_n) & \text{if } V_n \text{ is bounded,} \\ (2-g_n)/(1-g_n) & \text{if } V_n \text{ is not bounded;} \end{cases}$$

then $V \in$ AS. Moreover, the continued fraction (1.1) *converges absolutely in this case.*

PROOF. If some $a_p = 0$, since $B_p \neq 0$, for $p \geqq 1$, then (1.1) converges.

Hence we shall assume that $a_p \neq 0$ for $p \geqq 2$. Let

$$(3.1) \qquad h_1 = 0, \qquad h_p = -a_p B_{p-2}/B_{p-1} \qquad p \geqq 2,$$

then $T_p(h_p) = \infty$. Furthermore, since $T_p(h_p) = T_{p+1}(h_{p+1}) = T_p t_{p+1}(h_{p+1})$, then $h_p = t_{p+1}(h_{p+1})$ which implies that $h_{p+1} = -a_p/(1-h_p)$ for $p \geqq 1$. By (3.1) and Lemma 2.3, $|h_p| \leqq g_p$ if V_p is bounded and $|h_p| \geqq 2-g_p$ if V_p is not bounded. We now restate (1.4) in the following more useful form:

$$(3.2) \quad T_n(z) = f_{n-1} + (a_n A_{n-2} B_{n-1} - a_n B_{n-2} A_{n-1})/B_{n-1}^2(z - h_n).$$

Since $T_n(0) = f_{n-2}$ and $T_n(1) = f_n$ we see from (3.2) that

$$|(f_n - f_{n-1})/(f_{n-1} - f_{n-2})| = |(h_n/(1 - h_n)|.$$

Lane [2] proved that (1.1) converges absolutely, if there exists a sequence d of positive numbers such that, $|h_n/(1-h_n)| \leqq d_n/(1+d_{n+1})$ for $n \geqq 1$. Let $\mu_n = |h_n/(1-h_n)|$, then the sequence d will exist provided $\sum_{r=1}^{r=\infty} \mu_1\mu_2\mu_3 \cdots \mu_r$ converges. The maximum value of $|h_n/(1-h_n)|$ is $g_n/(1-g_n)$ if V_n is bounded and it is $(2-g_n)/(1-g_n)$ if V_n is not bounded. Hence the conditions of Theorem 3 are sufficient for the absolute convergence of (1.1).

THEOREM 4. *Suppose each of s and q is a positive number and $0<r<1$, and for $n \geqq 1$, $|a_{3n-1}| \geqq (1+q+s)^2$, $|a_{3n}| \leqq rq$, and $|a_{3n+1}| \leqq rs$. Then the sequence of approximants $f_2, f_3, f_5, f_6, \cdots, f_{3n-1}, f_{3n}, \cdots$ of* (1.1) *converges.*

LEMMA 4.1. *Suppose $a_2, a_3, a_4, \cdots$ is a sequence which satisfies the hypothesis of Theorem 4 and for $n \geqq 1$; Z_{3n-2}: $|z| \geqq s+q$, Z_{3n-1}: $|z-1| \leqq s+q$, and Z_{3n}: $|z-1| \leqq s/(s+q)$. Then $t_k(Z_k) \subset Z_{k-1}$ for $k \geqq 2$.*

A proof, similar to that of Lemma 2.1, is omitted. We note that 0 may be in Z_{3n-1}, and in this case $t_{3n-1}(0) = \infty$, a point in Z_{3n-2}.

LEMMA 4.2. *If* (1) *$a_2, a_3, a_4, \cdots$ is a sequence which satisfies the hypothesis of Theorem* 4, (2) *$Z_1, Z_2, Z_3, \cdots$ is a sequence which satisfies the hypothesis of Lemma* 4.1, *and* (3) *for $n \geqq 1$, $U_n = T_n(Z_n)$; then $U_{n+1} \subset U_n$.*

A proof, similar to that of Lemma 2.2 is omitted.

We note that $f_{3n} \in U_{3n}$ and $f_{3n-1} \in U_{3n-1}$, but f_{3n+1} is not in U_{3n+1} in case $s+q>1$.

LEMMA 4.3. *If $a_2, a_3, a_4, \cdots$ is a sequence which satisfies the hypothesis of Theorem* 4, *then for $n \geqq 1$;* (1) $|B_{3n-1}/B_{3n-2} - 1| \geqq (1+q+s)$, (2) $|B_{3n}/B_{3n-1} - 1| \leqq q/(q+s)$, *and* (3) $|B_{3n+1}/B_{3n} - 1| \leqq q+s$.

A proof, similar to that of Lemma 2.3, is omitted.

INDICATION OF A PROOF OF THEOREM 4. Methods similar to those used in the proof of Theorem 2 may be used to show that, for $n \geqq 1$, $R_{3n}/R_{3n-1} \leqq r$, where R_n is the radius of the region U_n. This is sufficient to show that the regions of the sequence U have only one point in common.

We note that if $s+q \leqq 1$ in Theorem 4, then $f_n \in U_n$ for every n, and hence (1.1) converges. This is a better result than can be obtained from Theorem 2, since the convergence of (1.1) can be obtained from Theorem 2 only in case $s+q<1$.

COROLLARY 4.1. *Suppose* $0<2s<1$, $0<r<1$, *and for each positive integer* n, V_{3n-1}: $|z| \geqq (1+2s)^2$, V_{3n}: $|z| \leqq rs$, *and* V_{3n+1}: $|z| \leqq rs$. *Then* V *is in* AS.

The regions in Corollary 4.1 are best in the sense that if $t>0$ and $0<c<4t(1+t)$ then the periodic continued fraction such that, for $n \geqq 1$, $a_{3n-1} = -(1+2t)^2+c$, $a_{3n}=a_{3n+1}=t$ diverges.

Cowling, Leighton and Thron [1] proved that, for $n \geqq 1$, $|a_{2n}| \leqq r^2$, $|a_{2n+1}| \geqq 2(r^2 - \cos \arg a_{2n+1}) + \delta$ for $\delta>0$ and $r>1$ is sufficient for convergence of (1.1).

The following argument shows that in the case of triple regions convergence is not obtained without suitable restrictions on the radii of the bounded regions.

THEOREM 5. *Suppose* $s>0$, V_{3n-1}: $|z| \geqq s$, V_{3n}: $|z| \leqq 1$, *and* V_{3n+1}: $|z| \leqq 1$. *Then there exists a sequence* $a_2, a_3, a_4, \cdots$ *such that*, $a_n \in V_n$ *for* $n \geqq 2$, *and* (1.1) *diverges.*

LEMMA 5.1. *The continued fraction* (1.1) *diverges in case* $a_p \neq 0$ *for* $p \geqq 2$, *and the following series converges*:

$$
\begin{aligned}
& |(1+a_2)/a_2| + |a_2(1+a_4)/a_3| + |a_3(1+a_5)/a_2a_4a_5| \\
(5.1) \qquad & + |a_2a_4a_5(1+a_7)/a_3a_6| + |a_3a_6(1+a_8)/a_2a_4a_5a_7a_8| \\
& + |a_2a_4a_5a_7a_8(1+a_{10})/a_3a_6a_9| + \cdots .
\end{aligned}
$$

For a proof see Wall [6].

PROOF OF THEOREM 5. Suppose t is a number greater than s and 1, and let $a_{3n-1}=t$, and $a_{3n}=a_{3n+1}=-1$ for $n \geqq 1$. In this case the series (5.1) reduces to

$$|t+1| \sum_{p=1}^{p=\infty} |1/t^p|.$$

Since this series converges for $t>1$, by Lemma 5.1 the continued fraction (1.1) diverges.

References

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University of Texas and
University of Kentucky

NOTE ON A NONLINEAR EIGENVALUE PROBLEM

MARVIN SHINBROT[1]

1. In the theory of hydrodynamic stability, eigenvalue problems of the form

$$Lu + \frac{1}{\lambda} Mu = \lambda u \tag{1.1}$$

arise [1, p. 430]. Here, L and M denote ordinary differential operators, the order of L exceeds that of M, and the boundary conditions are such that L is self-adjoint. One of the questions of interest is whether there exist eigenvalues of this problem and, if so, whether the corresponding eigenfunctions are complete.

Replacing λ by $1/\lambda$, it is easy to see that if L^{-1} exists, (1.1) is equivalent to

$$\lambda u = Au + \lambda^2 Bu, \tag{1.2}$$

where $A = L^{-1}$ and $B = -L^{-1}M$ are compact, and A is symmetric. In this note, we shall consider the question of the completeness of the eigenfunctions of the following generalization of (1.2):

$$\lambda u = Au + \lambda^\alpha B_\lambda u, \tag{1.3}$$

where $\alpha > 1$, A is compact and symmetric, and B_λ, which, as the notation indicates, may depend on λ, is merely bounded. More precise

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