SHORTER NOTE

The purpose of this department is to publish very short papers of an unusually elegant and polished character for which there is normally no other outlet.

THE CONVERSE OF MOORE'S GARDEN-OF-EDEN THEOREM

JOHN MYHILL[1]

We presuppose the terminology of Moore [1]. In this paper, Moore proves that the existence of two mutually erasable configurations in a tessellation universe is a sufficient condition for the existence of Garden-of-Eden configurations therein. We shall show that this condition[2] is both necessary and sufficient.

By an *environment* is meant a specification of states for all cells of the entire two-dimensional tessellation space with the exception of a square piece. By the *insertion* $E(C)$ of a configuration[3] C of appropriate size into an environment E is meant simply the result of specifying the states of the unspecified cells of E to be the states of the corresponding cells of C. By the *sequent* $E(C)'$ of C in E is meant the state of the universe at $t=1$, if $E(C)$ is the state of the universe at $t=0$. Two configurations C_1, C_2 of the same size are said to be *distinguished* by the environment E, if $E(C_1)' \neq E(C_2)'$.

Moore's argument shows that *if there are two configurations which cannot be distinguished, there are Garden-of-Eden configurations.* For the converse proposition suppose if possible that every pair of configurations can be distinguished, and that there exists a (square) Garden-of-Eden configuration G of side n. We easily establish the

Received by the editors May 29, 1962.

[1] Support by NSF grant G19001 is gratefully acknowledged.

[2] We are grateful to the referee for pointing out that the existence of two mutually erasable configurations in Moore's sense (op. cit.) is *equivalent* to the existence of two configurations which cannot be distinguished. The proof depends on the following easy strengthening of our Lemma: if every pair of distinct configurations can be distinguished by *some* environment, then every pair of configurations can be distinguished by *every* environment (of appropriate size).

[3] Our use of "configuration" differs slightly from Moore's in that we identify two copies of the same configuration if one is obtained from the other by a translation. However, we do *not* identify a configuration C with the result of surrounding it with a wall of one or more layers of blank cells: this convention is essential for understanding the proof of the Lemma.

LEMMA. *Any two configurations have distinct sequents in the environment E_0 consisting entirely of passive cells.*

For if the configurations C_1 and C_2 had identical sequents in E_0, the configurations C_1^* and C_2^*, obtained by adjoining to C_1 and C_2 a border of passive cells of width 2, would have identical sequents in *every* environment.

We infer immediately that for each number k, there are at least as many *sequent* (and consequently not Garden-of-Eden) $kn \times kn$ configurations, as there are $(kn-2) \times (kn-2)$ configurations altogether; i.e., at least $A^{(kn-2)^2}$ where A is the number of states.

On the other hand, there cannot be more than $(A^{n^2}-1)^{k^2}$ $kn \times kn$ configurations which do not contain a copy of G. Since every configuration which contains a copy of G is a Garden-of-Eden configuration, there are at most $(A^{n^2}-1)^{k^2}$ $kn \times kn$ configurations which are not Garden-of-Eden configurations. If ν is the number of such configurations we have

$$A^{(kn-2)^2} \leqq \nu \leqq (A^{n^2} - 1)^{k^2}$$

which, for large k, contradicts Moore's inequality (op. cit.)

$$(A^{n^2} - 1)^{k^2} < A^{(kn-2)^2}.$$

Thus we have proved that the existence of two indistinguishable configurations is a necessary as well as a sufficient condition for the existence of Garden-of-Eden configurations.

BIBLIOGRAPHY

1. E. F. Moore, *Machine models of self-reproduction*, Proc. Sympos. Appl. Math., Vol. 14, pp. 17–34, Amer. Math. Soc., Providence, R. I., 1963.

STANFORD UNIVERSITY