AN INTERPOLATION SERIES

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1. **Introduction.** Let $z_1, z_2, \cdots$ be a sequence of points in the unit disc, $0<|z_n|<1$, $z_n \neq z_m$ if $n \neq m$, $|z_1| \leqq |z_2| \leqq \cdots$ and clustering only at the boundary. In this note we shall construct an interpolation series, which generalizes the Taylor series, and we shall study two applications in opposite situations.

1°. It is very well known that $\lim_{n\to\infty} |z_1 \cdots z_n| = 0$ is a necessary and sufficient condition that a sufficiently bounded function holomorphic in the unit disc which vanishes at all z_n should vanish identically [4, Chapter VII]. Under a hypothesis on the z_n which is stronger than that implying unicity, we shall show that the interpolation series gives an effective representation of functions in the class H^1 interpolating arbitrary data in the points z_n.

2°. In the opposite case, when the sequence z_n is not a set of unicity, there exist interpolatory functions which possess an analytic continuation to regions much larger than the unit disc. We refer to [1] for the construction of such interpolations. The question then arises as to whether the particular interpolation given by some natural interpolation series also has this property. In Theorem 2 we shall establish that our interpolation series is capable of representing an interpolation in its maximal domain of existence—that is, outside its set of singularities. In this respect, the behavior is entirely different from that of Newton series, for example, which diverge outside of a certain lemniscate containing a singularity on its boundary.

2. **Statement of results.** We shall consider the interpolation problem $f(z_n) = w_n$, $n = 1, 2, \cdots$. Put

$$B_0(z) = 1, \qquad B_n(z) = \prod_{k=1}^{n} \frac{z - z_k}{1 - \bar{z}_k z} \qquad (n = 1, 2, \cdots).$$

Our results concern an interpolation series of the form

$$\sum_{0}^{\infty} c_n B_n(z); \tag{1}$$

this signifies that the coefficients c_n are linear forms in $w_1, \cdots, w_{n+1}$ determined by interpolating. The series (1) is a generalization of the Taylor series with which it coincides if the z_n are confounded with the origin.

Received by the editors May 5, 1962.

Recall that the class H^1 consists of all functions holomorphic in the unit disc such that the integral

$$\int_{-\pi}^{\pi} | f(re^{i\theta}) | \, d\theta$$

remains bounded as $r \to 1$. The boundary function $f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ then belongs to L^1.

Let us take a number θ, $0 < \theta < 1$, and put

$$p_n(z) = \theta \cdot \frac{1 - |z|^2}{\max\limits_{k \leq n} | z_k^* - z |^2}, \qquad z_k^* = \frac{1}{\bar{z}_k},$$

a quantity which is independent of n for z near the origin.

THEOREM 1. *If the interpolation problem*

$$f(z_n) = w_n, \qquad (n = 1, 2, \cdots)$$

can be solved with a function $f(z)$ belonging to the class H^1, then

$$f(z) = \sum_0^\infty c_n B_n(z); \tag{2}$$

the series is convergent for all $|z| < 1$ for which

$$\lim_{n\to\infty} \frac{| z_1 \cdots z_n |^{p_n(z)}}{1 - | z_{n+1} |} = 0. \tag{3}$$

It should be noted that the condition (3) implies that the interpolation must be unique in H^1.

On the other hand, if the interpolation $f(z_n) = w_n$ is not unique, one can still study the particular solution

$$\sum_0^\infty c_n B_n(z).$$

It is of interest that this series can converge at every point in the plane situated outside of the closure E of the sequence z_n^*.

THEOREM 2. *If*

$$\sum_1^\infty | w_n | < \infty \tag{4}$$

and if

$$\sup_n \prod_{k\neq n} \left| \frac{1 - \bar{z}_k z_n}{z_n - z_k} \right| < \infty, \tag{5}$$

then the interpolation series

$$\sum_0^\infty c_n B_n(z) \tag{6}$$

converges to a holomorphic function $f(z)$ *in the complement of the set* E, *in the complex plane. The function* $f(z)$ *satisfies* $f(z_n) = w_n$, $n = 1, 2, \cdots$, *and has simple poles at the points* z_n^*.

3. **Proof of Theorem 1.** A sequence of coefficients $\gamma_0, \gamma_1, \cdots$, depending upon t, is uniquely defined by the triangular systems

$$\gamma_0 B_0(z_k) + \cdots + \gamma_n B_n(z_k) = \frac{1}{t - z_k}, \qquad k = 1, 2, \cdots, n + 1.$$

As functions of t, the γ's are linear combinations of $1/(t - z_k)$. The function

$$1 - (t - z) \sum_{k=0}^{n} \gamma_k B_k(z), \qquad (t \neq z_k),$$

is necessarily of the form $A_n(t) B_n(z)(z - z_{n+1})$. Here $A_n(t)$ can be determined by putting $z = t$. In this way the identity in t, z results:

$$\frac{1}{t - z} - \sum_{k=0}^{n} \gamma_k(t) B_k(z) = \frac{B_n(z)(z - z_{n+1})}{B_n(t)(t - z_{n+1})} \frac{1}{t - z}.$$

Now if $f(z)$ is any function belonging to the class H^1, it is represented by the Cauchy integral over its boundary values. Hence, for $|z| < 1$, we obtain

$$f(z) - \sum_{k=0}^{n} c_k B_k(z) = \frac{1}{2\pi i} \int_{|t|=1} \frac{B_n(z)(z - z_{n+1})}{B_n(t)(t - z_{n+1})} \frac{f(t)}{t - z} dt, \tag{7}$$

$n = 0, 1, \cdots$, where

$$c_k = \int_{|t|=1} f(t) \gamma_k(t) dt. \tag{8}$$

It is evident from (7) that the coefficients c_k are uniquely determined from the equations

$$\sum_{k=0}^{n} c_k B_k(z_j) = w_j, \qquad j = 1, \cdots, n + 1.$$

Hence the series $\sum c_k B_k(z)$ is actually an interpolation series, a fact which might not be entirely obvious since the boundary values of f on $|t| = 1$ enter in the relation (8).

We proceed to the proof of Theorem 1, which is based on the identity (7). Let us put $\eta = \theta^{-1} > 1$. We can suppose that

$$-\eta(1 - |z_k|^2) < \log |z_k|^2, \qquad k = 1, 2, \cdots,$$

for this holds for all sufficiently large indices and it is only these which affect the convergence of the series (1). Writing

$$b_k = b_k(z) = \frac{z - z_k}{1 - \bar{z}_k z},$$

it then follows that

$$\begin{aligned} |B_n(z)| &= \exp\left(\frac{1}{2}\sum_1^n \log |b_k|^2\right) \\ &\leqq \exp\left\{-\frac{1}{2}\sum_1^n (1 - |b_k|^2)\right\} \\ &= \exp\left\{-\frac{1}{2}(1 - |z|^2)\sum_1^n \frac{1 - |z_k|^2}{|1 - \bar{z}_k z|^2}\right\} \\ &\leqq \exp\left\{-\frac{(1 - |z|^2)}{2\eta \max_{k \leqq n} |z_k^* - z|^2}\sum_1^n \eta(1 - |z_k|^2)\right\} \\ &< |z_1 \cdots z_n|^{p_n(z)}. \end{aligned}$$

In view of (3), (7) and the hypothesis that f belongs to H^1, it follows that the interpolation series (2) converges as asserted.

4. **Proof of Theorem 2.** The proof of Theorem 2 depends upon writing the interpolating function $\sum_0^n c_k B_k(z)$ as follows:

$$\begin{aligned} \sum_0^n c_k B_k(z) = {} & B_n(z) \sum_{k=1}^n (1 - |z_k|^2) C_{kn} \frac{w_k}{z - z_k} \\ & - B_n(z) \sum_{k=1}^n (1 - |z_k|^2) C_{kn} \frac{w_k}{z_{n+1} - z_k} + w_{n+1} \frac{B_n(z)}{B_n(z_{n+1})} \end{aligned} \tag{9}$$

where

$$C_{kn} = \prod_{j=1;\, j \neq k}^n \frac{1 - \bar{z}_j z_k}{z_k - z_j}.$$

This identity can be checked by evaluating the integral appearing in

(7) by residues; it is known [2] that on account of (4) and (5) a bounded interpolation exists in the unit disc, so that there is no difficulty in taking the Cauchy integral over $|t|=1$.

If we let M denote the finite supremum occurring in (5), then $|C_{kn}|\leqq M$. We also have

$$\left|\frac{(1-|z_k|^2)C_{kn}}{z_k-z_{n+1}}\right| = |C_{k,n+1}|\,\frac{1-|z_k|^2}{|1-\bar{z}_{n+1}z_k|}\leqq\frac{2M}{|z_1|},$$

and

$$|B_n(z_{n+1})| > \prod_{k\neq n+1}\left|\frac{z_{n+1}-z_k}{1-\bar{z}_k z_{n+1}}\right| \geqq \inf_n \prod_{k\neq n}\left|\frac{z_n-z_k}{1-\bar{z}_k z_n}\right| = \frac{1}{M} > 0.$$

The fact that $B_n(z)$ converges uniformly on compact sets situated outside of E, the preceding inequalities and hypothesis (4) together imply that the right-hand side of (9) converges as $n\to\infty$ uniformly on compact sets which do not intersect E and which do not contain any points z_k. However, in a small neighborhood of z_k we can take the factor $B_n(z)$ into the first summation on the right-hand side of (9) and use the boundedness of $B_n(z)/(z_k-z)$ to obtain the same convergence in this neighborhood. Therefore the series (6) represents a holomorphic function $f(z)$ outside of E. Obviously $f(z_n)=w_n$. That $f(z)$ has a simple pole at $z=z_n^*$ follows from the corresponding property of the finite Blaschke product $B_n(z)$ appearing in equation (9). This completes the proof of Theorem 2.

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