MULTIPLICATIVE FUNCTIONALS OF A MARKOV PROCESS

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The present note may best be viewed as an addendum to Meyer's important paper [2]. As such we refer the reader to [2] for all notations, definitions, etc. In particular $\{X_t\}$ will always denote a temporally homogeneous Markov process with state space X (locally compact, separable) satisfying Hunt's hypothesis (A). See [1] or [2].

Let $\{M_t\}$ be a normalized multiplicative functional of $\{X_t\}$ [2, p. 136 and p. 141], then Meyer [2, p. 153] has obtained the following very important result: $\{M_t\}$ has the strong Markov property, that is, for each stopping time T and random variable $R \geqq 0$ one has

(1.1) $$M_{R+T}(\omega) = M_T(\omega) M_R(\theta_T \omega)$$

a.s., P^x for each x in X. Unfortunately Meyer's proof contains a slight gap (in the proof of Theorem 4.2 on p. 152 of [2]) and the result is not valid without additional assumptions on $\{M_t\}$ as the following example shows. Let $\{X_t\}$ be one-dimensional Brownian motion (so that X is the real line) and define $M_t(\omega) = 1$ for all $t \geqq 0$ if $X_0(\omega) \neq 0$ and $M_t(\omega) = 0$ for all $t \geqq 0$ if $X_0(\omega) = 0$. It is easy to see that this defines a multiplicative functional of $\{X_t\}$ since $P^x(X_t = 0) = 0$ for all $t > 0$ and x in X. On the other hand if T is the first passage time to 0 it is immediate that (1.1) is not valid.

We state the following criterion for the strong Markov property.

THEOREM. *The normalized multiplicative functional $\{M_t\}$ has the strong Markov property if and only if for every stopping time T and x in X,*

(1.2) $$P^x[X_T \in N, M_T > 0] = 0,$$

where N is the (universally measurable) set of nonpermanent points of $\{M_t\}$, i.e., $N = \{x : P^x(M_0 = 0) = 1\}$.

PROOF. The sufficiency may be established exactly as in [2], the condition (1.2) being just the condition necessary to make the proof of Theorem 4.2 [2, p. 152] valid. (One can give a much simpler proof of the sufficiency using resolvents instead of semi-groups.) To prove the necessity let $R = \inf\{t : M_t = 0\}$, then R is a stopping time and the right continuity of $\{M_t\}$ implies that $P^x(M_R > 0, R < \infty) = 0$ for all x. Let T be any stopping time and consider the random variable

$H(\omega) = T(\omega) + R(\theta_T\omega)$. If x is fixed the strong Markov property yields

$$E^x[M_H, H < \infty] \leq E^x[M_T E^{X(T)}(M_R, R < \infty)] = 0.$$

Thus $H = T + R(\theta_T) \geq R$ on $\{H < \infty\}$ and so $T + R(\theta_T) \geq R$, both statements holding a.s., P^x. Therefore

$$P^x[X_T \in N, M_T > 0] = P^x[X_T \in N, T < R]$$
$$\leq E^x[P^{X(T)}(R > 0), X_T \in N]$$
$$= 0,$$

since $P^y (R > 0) = 0$ for all y in N.

If $T = \inf \{t > 0, X_t \in N\}$ is a stopping time, then (1.2) can be replaced by $P^x(M_T > 0, T < \infty) = 0$ for all x. This is the case if N is nearly analytic. The multiplicative functional defined in the second paragraph gives an example of a nonperfect multiplicative functional since any perfect multiplicative functional has the strong Markov property. (See [2, p. 136] for the definition of a perfect multiplicative functional.) As an example of a strongly Markov multiplicative functional for which the corresponding semi-group $\{Q_t\}$ is not exactly subordinate to $\{P_t\}$ (the semi-group of $\{X_t\}$), let $\{X_t\}$ be translation to right along the real line at unit speed. Put $M_t(\omega) = 0$ if $t + X_0(\omega) \geq 0$ and $X_0(\omega) \leq 0$, and $M_t(\omega) = 1$ otherwise. It is easily verified that this example has the desired properties.

REFERENCES

1. G. A. Hunt, *Markoff processes and potentials*. I, Illinois J. Math. **1** (1957), 44–93.

2. P. A. Meyer, *Fonctionelles multiplicatives et additives du Markov*, Ann. Inst. Fourier (Grenoble) **12** (1962), 125–230.

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