SOME ORTHOGONAL FUNCTIONS CONNECTED WITH POLYNOMIAL IDENTITIES. II[1]

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In an earlier paper [5] we proved a general polynomial identity, some special cases of which gave rise to sets of orthogonal functions [6]. In this paper we recast this general polynomial identity into such a form that it leads directly to a general class of orthogonal functions containing, as special cases, those given in [6] and also those introduced in [4].

Let $n_1, n_2, \cdots$ be a sequence of integers each $\geqq 2$ and let $p_0=1$ and $p_j=n_1n_2\cdots n_j$ for $j\geqq 1$. Then each integer n, $0\leqq n<p_m$, may be written uniquely in the form

$$n = a_0 + a_1p_1 + \cdots + a_{m-1}p_{m-1}, \qquad 0 \leqq a_j < n_{j+1}, \tag{1}$$

and we have

$$a_j = [n/p_j] - n_{j+1}[n/p_{j+1}]. \tag{2}$$

In [5] we proved the following: if $\sum_{n=0}^{n_j-1} f_j(n)n^t=0$ for $0\leqq t\leqq \alpha_j$ and P is any polynomial of degree $\leqq \alpha_1+\cdots+\alpha_m+m-1$ then

$$\sum_{n=0}^{p_m-1} \prod_{j=1}^{m} f_j([n/p_{j-1}] - n_j[n/p_j])P(x+n) = 0. \tag{3}$$

Since (3) holds for any integers $n_1, \cdots, n_m$ (each $\geqq 2$) it holds for $n_1', \cdots, n_m'$, where $n_j'=n_{m-j+1}$. Putting $p_j'=n_1'\cdots n_j'$ and writing $f_j'=f_{m-j+1}$ we see that the conditions leading to (3) become conditions leading to

$$\sum_{n=0}^{p_m'-1} \prod_{j=1}^{m} f_j'([n/p_{j-1}'] - n_j'[n/p_j'])P(x+n) = 0$$

for the same polynomials P. Noting that $p_j'=p_m/p_{m-j}$ we have the following: if $\sum_{n=0}^{n_j-1} f_j(n)n^t=0$ for $0\leqq t\leqq \alpha_j$ and P is any polynomial of degree $\leqq \alpha_1+\cdots+\alpha_m+m-1$ then

$$\sum_{n=0}^{p_m-1} \prod_{j=1}^{m} f_j([np_j/p_m] - n_j[np_{j-1}/p_m])P(x+n) = 0. \tag{4}$$

There are p_m coefficients in this identity and we define $G_m(x)$ to be the periodic step function, with period unity, taking the value of

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the ith coefficient in (4) on the ith subinterval of length $1/p_m$ in $[0, 1)$, it is easy to see that

$$G_m(x) = \prod_{j=1}^{m} f_j([p_j x] - n_j[p_{j-1}x]). \tag{5}$$

Such a function is defined for each natural number m.

For convenience in our future expressions we introduce two other sequences of functions of x suggested by (5).

$$\begin{aligned} \nu_j(x) &= [p_j x] - n_j[p_{j-1}x] \\ \phi_{j-1}(x) &= f_j(\nu_j(x)) \end{aligned} \qquad \text{for } j = 1, 2, \cdots. \tag{6}$$

It is clear that $G_m(x) = \prod_{j=0}^{m-1} \phi_j(x)$ and

$$\nu_j(x + 1/p_{j-1}) = \nu_j(x), \qquad j = 1, 2, \cdots. \tag{7}$$

The functions $\nu_j(x)$ have a more obvious interpretation than that given in the above context. Indeed, these functions are merely the digits in the Cantor expansion (see [3, p. 7]) of $x - [x]$ relative to the p_j.

$$x - [x] = \nu_1(x)/p_1 + \nu_2(x)/p_2 + \cdots. \tag{8}$$

Writing $\mu\{\nu_i \leqq d_i, \cdots, \nu_h \leqq d_h\}$ for the Lebesgue measure of the set of x, $0 \leqq x < 1$, for which $\nu_i(x) \leqq d_i, \cdots, \nu_h(x) \leqq d_h$ we see that

$$\begin{aligned} \mu\{\nu_1 \leqq d_1, \cdots, \nu_m \leqq d_m\} &= \sum \mu\{\nu_1 = a_1, \cdots, \nu_m = a_m\} \\ &= (d_1 + 1)(d_2 + 1) \cdots (d_m + 1)/p_m, \end{aligned}$$

where the sum is taken over all m-tuples $(a_1, \cdots, a_m)$ for which $0 \leqq a_j \leqq d_j$. Also

$$\begin{aligned} \mu\{\nu_j \leqq d_j\} &= \mu\{\nu_1 \leqq n_1 - 1, \cdots, \nu_{j-1} \leqq n_{j-1} - 1, \nu_j \leqq d_j\} \\ &= n_1 \cdots n_{j-1}(d_j + 1)/p_j \\ &= (d_j + 1)/n_j. \end{aligned}$$

Therefore

$$\mu\{\nu_1 \leqq d_1, \cdots, \nu_m \leqq d_m\} = \mu\{\nu_1 \leqq d_1\} \cdots \mu\{\nu_m \leqq d_m\}; \tag{9}$$

i.e., the ν_j functions are statistically independent (see [2]).

Using (9) we give an easy proof of

$$\int_0^1 \prod_{j=0}^{s-1} \phi_j^{\beta_j}(x)dx = \prod_{j=0}^{s-1} \int_0^1 \phi_j^{\beta_j}(x)dx \left(= \prod_{j=0}^{s-1} \sum_{n=0}^{n_j-1} f_{j+1}^{\beta_j}(n) \right). \tag{10}$$

The second equality in (10) is clear and the first goes as follows.

$$\int_0^1 \prod_{j=0}^{s-1} \phi_j^{\beta_j}(x)dx = \int_0^1 \prod_{j=1}^{s} f_j^{\beta_j-1}(\nu_j(x))dx$$

$$= \sum_{a_1,\cdots,a_s;\, 0\leq a_j<n_j} \prod_{j=1}^{s} f_j^{\beta_j-1}(a_j)\mu\{\nu_1 = a_1, \cdots, \nu_s = a_s\}$$

$$= \sum_{a_1,\cdots,a_s;\, 0\leq a_j<n_j} \prod_{j=1}^{s} (f_j^{\beta_j-1}(a_j)\mu\{\nu_j = a_j\})$$

$$= \prod_{j=1}^{s} \int_0^1 f_j^{\beta_j-1}(\nu_j(x))dx = \prod_{j=0}^{s-1} \int_0^1 \phi_j^{\beta_j}(x)dx.$$

When the f_j satisfy the conditions leading to (4) the identity (10) guarantees the vanishing of the integral of power products of the ϕ_j functions in which at least one exponent β_j is unity. In particular the functions $G_m(x)$ in (5) are orthogonal.

In the special case

$$f_j(n) = (-1)^n \binom{n_j - 1}{n}$$

equation (4) becomes

$$(11) \quad \sum_{n=0}^{p_m-1} (-1)^{a_0+\cdots+a_{m-1}} \binom{n_m - 1}{a_0} \cdots \binom{n_1 - 1}{a_{m-1}} P(x+n) = 0$$

for P a polynomial of degree $\leqq n_1 + \cdots + n_m - m - 1$ and where, the a_i are defined by

$$n = a_0 + a_1 p_m/p_{m-1} + a_2 p_m/p_{m-2} + \cdots + a_{m-1} p_m/p_1, \quad 0 \leqq a_j < n_{m-j}.$$

The corresponding functions G_m, given in (5), are orthogonal and in the case where all $n_j = b \geqq 2$ are the functions t_n defined in [6]. In this case also the s_n functions of that paper are the ϕ_j functions here. Thus Theorem 3 of that paper, dealing with power products of the s_n functions, is contained in (10).

In the special case $f_j(n) = \exp(2\pi n i/n_j)$ we have

$$\phi_{j-1}(x) = f_j(\nu_j(x)) = \exp(2\pi([p_j x] - n_j[p_{j-1}x])i/n_j) = \exp 2\pi[p_j x]i/n_j.$$

These are the orthogonal functions ϕ_n defined in [4]. Again (10) furnishes us with a proof of the orthogonality of power products of these functions. Indeed, if Φ_n and Φ_m are two such power products then $\Phi_n\bar{\Phi}_m$ is of the form $\phi_0^{\beta_1} \cdots \phi_k^{\beta_{k+1}}$, where $0 \leqq \beta_j < n_j$, and not all $\beta_j = 0$ when $n \neq m$. We now need only observe that

$$\sum_{n=0}^{n_j-1} (\exp(2\pi n i/n_j))^{\beta_j} = 0 \qquad \text{for } 0 < \beta_j < n_j.$$

In particular when all n_i are equal to b the ϕ_j are the Rademacher and the generalized Rademacher functions [1] in the cases where $b=2$ and $b>2$, respectively.

References

1. H. E. Chrestensen, *A class of generalized Walsh functions*, Pacific J. Math. 5 (1955), 17–31.

2. Mark Kac, *Statistical independence in probability, analysis and number theory*, Carus Math. Monographs, No. 12, Wiley, New York, 1959.

3. Ivan Niven, *Irrational numbers*, Carus Math. Monographs, No. 11, Wiley, New York, 1956.

4. J. J. Price, *Certain groups of orthonormal step functions*, Canad. J. Math. 9 (1957), 413–425.

5. J. B. Roberts, *Polynomial identities*, Proc. Amer. Math. Soc. 11 (1960), 723–730.

6. ———, *Some orthogonal functions connected with polynomial identities*. I, Proc. Amer. Math. Soc. 9 (1958), 364–370.

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