AN ARITHMETIC PROPERTY OF RIEMANN SUMS[1]

WALTER RUDIN

If f is a real function on the real line, periodic with period 1, define

$$(1) \qquad (M_n f)(x) = \frac{1}{n}\sum_{i=1}^{n} f\left(x + \frac{i}{n}\right) \qquad (n = 1, 2, 3, \cdots).$$

Writing $\int f$ for the integral of f over $[0, 1]$, the relation

$$(2) \qquad \lim_{n\to\infty} (M_n f)(x) = \int f$$

holds for all real x if f is Riemann integrable on $[0, 1]$. In the present note it is shown that there are bounded measurable functions f for which (2) is false for every x and that this convergence problem has some interesting number-theoretic aspects.

In 1934, Jessen [1] proved that if $f \in L^1$ on $[0, 1]$ and if $\{n_k\}$ is an increasing sequence of positive integers in which *each term divides the next, then*

$$(3) \qquad \lim_{k\to\infty} (M_{n_k} f)(x) = \int f$$

for almost all x.

In 1948, Salem [2] showed that (3) holds for almost all x if the integral modulus of continuity of f satisfies a certain condition and if $\{n_k\}$ satisfies a corresponding lacunarity condition. Salem's condition involves only the rate of growth of $\{n_k\}$; no divisibility assumptions appear.

In the opposite direction, it is known (I am indebted to the referee for mentioning [4] and [5]) that there are functions $f \in L^1$ for which (2) fails almost everywhere. For example, if $0 < \alpha < 1/2$, define

$$(4) \qquad f(x) = |x|^{-1+\alpha} \qquad (|x| \leqq 1/2)$$

and define $f(x)$ for all other x by periodicity. For every irrational x, there are infinitely many integers n such that

$$(5) \qquad \left| x - \frac{m}{n} \right| < \frac{1}{n^2}$$

Presented to the Society, February 25, 1963, under the title *Prime numbers and measurable functions*; received by the editors February 8, 1963.

[1] This research was supported by NSF grant GP-249.

for some integer m; if (5) holds, then $f(x-m/n)>f(1/n^2)=n^{2-2\alpha}$ so that $(M_nf)(x)>n^{1-2\alpha}$. Thus

$$\limsup_{n\to\infty} (M_nf)(x) = +\infty \tag{6}$$

for almost all x. If $p<2$, we may choose α so that $1-1/p<\alpha<1/2$, and thus get examples of $f\in L^p$ for which (6) holds almost everywhere.

This crude method does not settle the problem for L^2, nor, a fortiori, for bounded measurable functions. However, (2) fails even there, and it turns out that arithmetic properties of $\{n_k\}$ *are* crucial; Remark (A) at the end of this note makes this very evident.

THEOREM. *Let S be a sequence of positive integers which contains sets S_N ($N=1, 2, 3, \cdots$), each consisting of N terms, such that no member of S_N divides the least common multiple of the other members of S_N.*

Then to every $\epsilon>0$ there exists a bounded measurable function f, periodic with period 1, such that $0\leqq f\leqq 1$, and such that

$$\limsup_{n\in S} (M_nf)(x) \geqq \tfrac{1}{2}$$

for all x, although $\int f<\epsilon$.

For instance, S could be any sequence of primes.

In the proof, f is constructed as the characteristic function of an open set.

PROOF. We may assume, without loss of generality, that the sets S_N are pairwise disjoint.

Fix $N>2$, choose $\delta>0$ such that $\delta^N=N^{-1}(\log N)^{-2}$, let g and h be the characteristic functions of sets G and H, where G is the union of the segments $(k, k+\delta)$ ($k=0, \pm 1, \pm 2, \cdots$), and H is the complement of G. If $n_1, \cdots, n_N$ are the members of S_N, let k_i be the least common multiple of $n_1, \cdots, n_{i-1}, n_{i+1}, \cdots, n_N$, for $i=1, \cdots, N$. We can then find integers $p_1, \cdots, p_N$, each so much larger than the preceding one that the following is true: if

$$\phi_N(t) = \prod_{i=1}^{N} g(k_ip_it),$$

$$\psi_{j,N}(t) = h(k_jp_jt) \prod_{i\neq j} g(k_ip_it) \qquad (1 \leqq j \leqq N),$$

then $\int\phi_N$ and $\int\psi_{j,N}$ differ by as little as we please from the products

$$\left(\int g\right)^N = \delta^N \tag{7}$$

and

$$\left(\int h\right)\left(\int g\right)^{N-1} = (1-\delta)\delta^{N-1}. \tag{8}$$

Put $A_N = B_{1,N} \cup \cdots \cup B_{N,N}$, where $B_{j,N}$ is the set whose characteristic function is $\psi_{j,N}$. Since the sets $B_{1,N}, \cdots, B_{N,N}$ are pairwise disjoint, and since

$$N(1-\delta)\delta^{N-1} = N\delta^N\left(\frac{1}{\delta} - 1\right) > N\delta^N(N^{1/N} - 1) > \delta^N \log N,$$

we see from (7) and (8) that $p_1, \cdots, p_N$ can be so chosen that

$$\int \phi_N < \frac{2}{N \log^2 N}, \qquad m(A_N) > \frac{1}{N \log N}, \tag{9}$$

where $m(A_N)$ denotes the Lebesgue measure of $A_N \cap [0, 1]$. Moreover, $p_1, \cdots, p_N$ can be chosen to be primes which divide no member of S_N.

Suppose now that $x \in B_{j,N}$. Then $k_i p_i x \in G$ if $i \neq j$, and $k_j p_j x \in H$. Since n_j divides k_i if $i \neq j$, we have

$$k_i p_i\left(x + \frac{r}{n_j}\right) \equiv k_i p_i x \pmod 1 \tag{10}$$

if $i \neq j$ and $r = 1, \cdots, n_j$. But n_j does not divide $k_j p_j$ (this is where the arithmetic hypothesis imposed on S_N is used), and therefore the terms of the arithmetic progression

$$k_j p_j\left(x + \frac{r}{n_j}\right) \qquad (r = 1, \cdots, n_j) \tag{11}$$

are not all congruent (mod 1); since $\delta > 1/2$, it follows that at least half of them lie in G. Hence

$$(M_{n_j}\phi_N)(x) \geqq \tfrac{1}{2} \tag{12}$$

for all $x \in B_{j,N}$, and (12) implies that

$$\max_{n \in S_N} (M_n \phi_N)(x) \geqq \tfrac{1}{2} \qquad (x \in A_N). \tag{13}$$

By (9), $\sum m(A_N) = \infty$. Hence there is a sequence $\{\alpha_N\}$ of real numbers such that almost every x lies in infinitely many of the translates $A_N + \alpha_N$ (see [**3**, p. 165, Lemma 1.24]). Choose N_0 so that

$$4 \sum_{N_0}^{\infty} N^{-1}(\log N)^{-2} < \epsilon \tag{14}$$

and put

$$\phi(t) = \sup_{N \geq N_0} \phi_N(t - \alpha_N). \tag{15}$$

Then (13) implies

$$\limsup_{n \in S} (M_n\phi)(x) \geq \tfrac{1}{2} \tag{16}$$

for almost all x, although $\int\phi < \epsilon/2$, by (9) and (14).

If now E is the set of measure 0 on which (16) fails, let χ be the characteristic function of a periodic open set V which contains $E+r$ for all rational numbers r, and such that $m(V) < \epsilon/2$. Setting $f = \max(\phi, \chi)$, we obtain a function which has the properties asserted by the theorem.

REMARKS. (A) There are sequences $\{n_k\}$ which satisfy the hypothesis of Jessen's theorem but such that $\{1+n_k\}$ is a sequence of primes. To see this, suppose n_k is chosen; by Dirichlet's theorem on primes in arithmetic progressions, there is an integer $r > 1$ such that $q = 1 + rn_k$ is prime; put $n_{k+1} = rn_k$.

Thus $\{(M_{n_k}f)(x)\}$ *converges to* $\int f$ *a.e. although the sequence* $\{(M_{1+n_k}f)(x)\}$ *need not do so.*

(B) Take $\epsilon_k = 2^{-k}$ in our theorem, let f_k ($f = 1, 2, 3, \cdots$) be the corresponding functions, and put $F = \sum kf_k$. Then $F \in L^p$ on $[0, 1]$ for every $p < \infty$, but

$$\limsup_{n \in S} (M_nF)(x) = +\infty \tag{17}$$

for every x.

(C) It is easy to see that $M_nf \to f$ in the L^p-norm, as $n \to \infty$, for every $f \in L^p$, if $1 \leq p < \infty$. Hence for every $f \in L^1$ there is a sequence $\{n_k\}$ such that (3) holds almost everywhere.

REFERENCES

1. B. Jessen, *On the approximation of Lebesgue integrals by Riemann sums*, Ann. of Math. (2) **35** (1934), 248–251.

2. R. Salem, *Sur les sommes Riemanniennes des fonctions sommables*, Mat. Tidsskr. B **1948**, 60–62 (1948).

3. A. Zygmund, *Trigonometric series*, 2nd ed., Vol. II, Cambridge Univ. Press, Cambridge, 1959.

4. J. Marcinkiewicz and A. Zygmund, *Mean values of trigonometrical polynomials*, Fund. Math. **28** (1937), 131–166.

5. H. D. Ursell, *On the behaviour of a certain sequence of functions derived from a given one*, J. London Math. Soc. **12** (1937), 229–232.

UNIVERSITY OF WISCONSIN