AN ISOPERIMETRIC INEQUALITY

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If f maps the k-cube I^k into R^n so that the images of opposite $k-1$ faces of I^k are at least distance b apart, it is plausible that the volume of f can be no smaller than b^k. This is true if f is Lipschitzian and is established as a corollary to the more general theorem below. This problem was suggested by D. C. Spencer, who together with S. Bergman proved it for $k=2$ in [2].

DEFINITIONS.

(1) $I_*(R^n) = \oplus_j I_j(R^n)$ is the chain complex of integral currents in R^n with boundary homomorphism ∂ as defined in [3]. The mass of a current T is written $M(T)$. For $T \in I_0(R^n)$ let $M_0(T)$ be the absolute value of the coefficient sum of T.

(2) $\mathfrak{g}^k$ is the cell complex of the k-cube I^k [1; 2.1], i.e., the chain complex generated by the cubical faces of I^k of various dimensions. $\mathfrak{g}^1$ is generated by $\{[0, 1], [0], [1]\}$ with $d[0, 1]=[1]-[0]$, $d[0]=d[1]=0$. $\mathfrak{g}^2$ is generated by $\{[0, 1]\otimes[0, 1], [0, 1]\otimes[1], [0]\otimes[0], \cdots\}$, etc. Let $\alpha^k=[0, 1]\otimes \cdots \otimes[0, 1]$ be the unique k cell in $\mathfrak{g}^k$, and let $\alpha^k(i, \epsilon)$, $i=1, 2, \cdots, k$, $\epsilon=0, 1$ denote the $k-1$ cell obtained by setting the ith coordinate equal to ϵ.

(3) For $A \subset R^n$ define $u_A: R^n \to R$, $u_A(x)=\text{distance }(x, A)$. Set $U_r = R^n \cap \{x: u_A(x) < r\}$ for $r \in R$. Note that u_A satisfies a Lipschitz condition with constant 1 for any A.

THEOREM. *Let F: $\mathfrak{g}^k \to I_*(R^n)$ be a chain map of degree 0 such that*

(1) *For some 0 cell (vertex) $v \in \mathfrak{g}^k$, $M_0(F(v)) \geqq 1$.*

(2) $\inf\{|x-y| : x \in$ *support* $(F(\beta))$, $y \in$ *support* $(F(\gamma))\} \geqq b_i$ *whenever β is a face of $\alpha^k(i, 0)$ and γ is a face of $\alpha^k(i, 1)$ for some $i=1, 2, \cdots, k$.*

Then $M(F(\alpha^k)) \geqq M_0(F(v)) \cdot \prod_{i=1}^k b_i$.

LEMMA. *Let A and B be subsets of R^n with* $\inf\{|x-y| : x \in A, y \in B\} = b$. *Suppose S, $T \in I_0(R^n)$, support $(S) \subset A$, support $(T) \subset B$, and $M_0(S) \geqq 1$. Suppose also $Q \in I_1(R^n)$ with $\partial Q = S - T$. Then $M_0(S) = M_0(T)$ and $M(Q) \geqq M_0(S) \cdot b$. Also, for L_1 almost all $r \in (0, b)$, $\partial(Q \cap U_r(A)) - \partial Q \cap U_r(A) \in I_0(R^n)$ and*

$$M_0(\partial(Q \cap U_r(A)) - \partial Q \cap U_r(A)) = M_0(S).$$

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PROOF. The lemma follows in part from [3, 3.8(3), 3.9, 3.10, 8.14] and the rest is obvious.

PROOF OF THE THEOREM. By induction. For $k=1$, the theorem is implied by the lemma. Assume the theorem holds up to dimension $k-1$. Consider the map

$$G_r: I^{k-1} \to I_*(R^n)$$
$$G_r(\beta) = \partial[F([0,1] \otimes \beta) \cap U_r(\text{support}[F(\alpha^k(1,0))])] - \partial F([0,1] \otimes \beta) \cap U_r(\text{support}[F(\alpha^k(1,0))])$$

for each cell $\beta \in \mathcal{G}^{k-1}$. By [3; 3.8(3), 3.9, 3.10, 8.14], G_r is defined for L_1 almost all $r \in (0, b_1)$. One verifies that for each such r, G_r is a chain map. Using the lemma, it follows that each such G_r satisfies the induction hypothesis. Thus

$$M(G_r(\alpha^{k-1})) \geqq M_0(F(v)) \cdot \prod_{i=2}^{k} b_i.$$

Using [3, 3.10] one sees

$$M(F(\alpha^k)) \geqq \int_0^{b_1} M(G_r(\alpha^{k-1}))dr \geqq M_0(F(v)) \cdot \prod_{i=1}^{k} b_i.$$

COROLLARY. *Let $f: I^k \to R^n$ be Lipschitzian with $|f(x)-f(y)| \geqq b$ whenever x and y lie on opposite $k-1$ dimensional faces of I^k. The restrictions of f to the various faces of I^k determine integral currents in R^n, and since volume is always greater than or equal to mass, one has*

$$\text{volume}(f) \geqq M(f_\#(I^k)) \geqq b^k.$$

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