OBSTRUCTIONS TO EXTENDING DIFFEOMORPHISMS

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In [2], we defined a theory of obstructions to constructing a diffeomorphism between two combinatorially equivalent differentiable manifolds. In the present paper, we apply this theory to the problem of extending a diffeomorphism of the boundaries of two manifolds to a diffeomorphism of the manifolds. A corollary is a theorem of R. Thom concerning the equivalence of differentiable structures.

All differentiable manifolds and maps are assumed to be of class C^∞; a *combinatorial equivalence* between two differentiable manifolds is an isomorphism between smooth (C^∞) triangulations of them.

THEOREM 1. *Let M and N be combinatorially equivalent differentiable n-manifolds. Let* $\mathrm{Bd}M$ *be the disjoint union of M_0 and M_1, where each is a union of components of* $\mathrm{Bd}M$; *similarly, let* $\mathrm{Bd}N = N_0 \cup N_1$. *Let $f: M_0 \to N_0$ be a diffeomorphism which is extendable to a combinatorial equivalence $F: M \to N$. The obstructions to extending f to a diffeomorphism of M onto N are elements of* $\mathcal{H}_m(M, M_1; \Gamma^{n-m})$; *if these obstructions vanish, f may be so extended.*

(Here $\mathcal{H}_m$ denotes homology based on infinite chains; m ranges from 0 to n; M_1 may be empty. The coefficient group Γ^{n-m} is defined in [2]; the coefficients are twisted if M is nonorientable.)

PROOF. We assume, without loss of generality, that the restriction of F to some neighborhood of M_0 is a diffeomorphism. (One may justify this as follows: Consider the manifold obtained from M and $M_0 \times I$ by identifying $x \in M_0$ with $(x, 0) \in M_0 \times I$; it is diffeomorphic to M. Obtain a manifold from N and $N_0 \times I$ similarly. The map which equals F on M and equals the trivial extension of f on $M_0 \times I$ is a combinatorial equivalence between these manifolds, and its restriction to $M_0 \times (0, 1]$ is a diffeomorphism.)

Restrict F to the manifold $M' = M - M_0$; it will be a combinatorial equivalence between this manifold and $N' = N - N_0$. We apply our obstruction theory (in particular, 5.7 of [2]) to the problem of smoothing this map; our object is to obtain a diffeomorphism of M' onto N' which equals F in a neighborhood of M_0. Such a diffeomorphism will give the required extension of f.

Now $F: M' \to N'$ is a diffeomorphism mod the $(n-1)$-skeleton of M'; we denote it by F_{n-1}. As an induction hypothesis, assume F_m is a

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diffeomorphism mod the m-skeleton of M', which equals F in a neighborhood of M_0. Let us suppose for the moment that the obstruction $\lambda_m F_m$ is homologous to zero, mod $\operatorname{Bd} M' = M_1$; let c be the chain it bounds. We may modify F_m near the carrier of c, obtaining a diffeomorphism mod m, $F'_m\colon M' \to N'$, such that $\lambda_m F'_m = 0$. Then F'_m may be smoothed to F_{m-1}, a diffeomorphism mod $m-1$. Now the carrier of $\lambda_m F_m$ is disjoint from some neighborhood of M_0, because F_m is already a diffeomorphism in some such neighborhood. If the carrier of c has the same property, then $F'_m = F_m = F$ in some such neighborhood. It follows that F_{m-1} may be chosen equal to F'_m in some such neighborhood, since F'_m is already differentiable there. Providing these suppositions hold at each stage of the induction, the smoothing process can be continued until one obtains a diffeomorphism which equals F in a neighborhood of M_0.

From the preceding analysis, it is clear that the homology class of our obstruction lies in those homology groups of M', mod M_1, which are based on chains whose carriers have no point of M_0 as a limit point. (The chains may be infinite, however.) These groups are easily proved to be isomorphic with the groups $\mathcal{H}_m(M, M_1; \Gamma^{n-m})$; the isomorphism is in fact induced by the inclusion of M' into M.

THEOREM 2 (THOM [3]). *Let M be a differentiable manifold whose boundary has two components M_0 and M_1; suppose M is combinatorially equivalent to the differentiable manifold $P \times I$. If M_0 is diffeomorphic to P, so is M_1, and M is diffeomorphic to $P \times I$.*

PROOF. Let $f\colon M_0 \to P \times 0$ be a diffeomorphism. If we can extend f to a combinatorial equivalence of M onto $P \times I$, then the theorem follows, for all the obstructions lie in $\mathcal{H}_m(M, M_1; \Gamma^{n-m})$, and these groups vanish because M_1 is a deformation retract of M. Let K be a complex which serves to triangulate P smoothly. Then any simplicial subdivision of the cell-complex $K \times I$ serves to triangulate $P \times I$ smoothly. Because M is combinatorially equivalent to P, any two smooth triangulations of M and $P \times I$ have isomorphic subdivisions ([4]; see also [1]). In particular, there is a smooth triangulation $h\colon L \to M$ such that L is a simplicial subdivision of $K \times I$. Assume h carries $|K \times 0|$ onto M_0, for convenience.

Let $K' \times 0$ be the subdivision of $K \times 0$ induced by L. Then $K' \times I$ is a cell-subdivision of $K \times I$, as is L; they have a common simplicial subdivision L'.

$$L' \xrightarrow{h} M,$$
$$K' \times 0 \xrightarrow{h_0} M_0 \xrightarrow{f} P.$$

Let h_0 be the restriction of h to $|K\times 0|$, then fh_0: $K'\times 0\rightarrow P$ is a smooth triangulation. Let F: $|K\times I|\rightarrow P\times I$ be the trivial extension of fh_0; because L' is a subdivision of $K'\times I$, F will be smooth on each simplex of L'. Hence Fh^{-1}: $M\rightarrow P\times I$ is a combinatorial equivalence; its restriction to M_0 is f, as desired.

References

1. J. R. Munkres, *Elementary differential topology*, Annals of Mathematics Studies No. 54, Princeton Univ. Press, Princeton, N. J., 1963.

2. ———, *Obstructions to the smoothing of piecewise-differentiable homeomorphisms*, Ann. of Math. (2) **72** (1960), 521–554.

3. R. Thom, *Des variétés triangulées aux variétés différentiables*, Proc. Internat. Congr. Math., 1958, pp. 248–255, Cambridge Univ. Press, Cambridge, 1960.

4. J. H. C. Whitehead, *On C^1-complexes*, Ann. of Math. (2) **41** (1940), 809–824.

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