PROJECTIONS ONTO CONTINUOUS FUNCTION SPACES

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1. **Introduction.** Goodner [3] introduced the family P_λ ($\lambda \geqq 1$) of Banach spaces X with the *λ-projection property*: For every imbedding of X as a subspace of a Banach space Z, there exists a projection P of Z onto X with $\|P\| \leqq \lambda$. The space $M(S)$ of all bounded real (or complex) valued functions on a set S, with the supremum norm, is a P_1 space, as can be seen by pointwise application of the Hahn-Banach theorem. X is a P_λ space for some finite λ if and only if it is a direct factor in some $M(S)$ space.

A complete characterization of the P_1 spaces is known (Kelley [7], Hasumi [5]): X is a P_1 space if and only if it is isometric to a space $C(S)$ of all continuous (real or complex) functions on an extremally disconnected compact Hausdorff space S. (A topological space is called extremally disconnected if the closure of every open set is open.) If X is isomorphic to a P_1 space it is a P_λ space. The open question whether these are the only P_λ spaces seems to be difficult. Some necessary conditions for X to be a P_λ space are known. We mention the following (Grothendieck [4]): (G) If X is a P_λ space then every weakly* convergent sequence in X^* is weakly convergent.

Related to the P_λ spaces are the P'_λ spaces, i.e., separable Banach spaces X with the *separable λ-projection* property: For every imbedding of X as a subspace of a separable Banach space Z, there exists a projection P of Z onto X with $\|P\| \leqq \lambda$. Sobczyk [10] proved that c_0 is a P'_2 space. All known P'_λ spaces are isomorphic to c_0.

In the present paper we investigate P_λ and P'_λ spaces which are continuous function spaces. S will denote a compact Hausdorff space, and $C(S)$ the Banach space of all continuous real-valued functions on S with the maximum norm. (With slight modifications all our results can be extended to the complex case.)

Some previous results (Isbell and Semadeni [6], Amir [1]) are:

A. If $C(S)$ is a P_λ space then every convergent sequence in S is eventually constant.

B. Let $i(S)$ be the maximal cardinal of families of open disjoint sets in S whose closures have a nonempty intersection. If $i(S) \geqq n$

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then there exists a Banach space Z such that $C(S)$ is a subspace of Z (with deficiency $n-1$) and every projection of Z onto $C(S)$ has a norm not smaller than $3-2/n$.

Our main tool in this paper is a generalization of B (Theorem 1). It is used to obtain the results:

C. (Theorem 2). If $C(S)$ is a P_λ space then S contains a dense, open, and extremally disconnected subspace. (This theorem was obtained in [1] by a more complicated method.)

D. (Theorem 3). If $C(S)$ is a P'_λ space of an infinite dimension, then it is an isomorph of c_0.

Theorem 2 solves in the negative problem 5 in [6]: If X is an infinite discrete space, then $C(\beta X-X)$ is not a P_λ space (Corollary 2), although it is a continuous image of $C(\beta X)$ which is a P_1 space.

Theorem 1 enables us to construct P_λ spaces of the $C(S)$ type with the exact projection constant $\lambda=1+2\rho$ for each real ρ belonging to the closure of the set

$$\left\{\sigma;\ \sigma = \sum_{i=1}^{k}\left(1-\frac{1}{n_i}\right);\ k,\ n_i \text{ natural}\right\}$$

(Remark c, §3).

2. **Projections from $B(S, \Sigma)$ onto $C(S)$.** Let S be a compact Hausdorff space. Let Σ be a field of subsets of S which contains an open basis and is closed under complementation, finite union and the closure operation. $B(S, \Sigma)$ is defined as the closed subspace spanned in $M(S)$ by the characteristic functions of the sets in Σ; $B(S, \Sigma)$ contains $C(S)$ as a subspace.

Definition. Let $\rho_1(s, \Sigma)=\sup_n\{1-1/n\}$, where the supremum is taken over all n such that there exist n open disjoint sets $G_1, \cdots, G_n \in\Sigma$ with $s\in\bigcap_{i=1}^n \bar{G}_i$.

For $k>1$ let

$$\rho_k(s, \Sigma) = \sup\left\{\left(1-\frac{1}{n}\right) + \inf_{u}\ \min_{i=1,\cdots,n}\ \sup_{t\in G_i\cap u} \rho_{k-1}(t, \Sigma)\right\},$$

where the first supremum is taken over all finite families $\{G_1, \cdots, G_n\}$ of disjoint open sets in Σ, such that $s\in\bigcap_{i=1}^n \bar{G}_i$, and the infimum is taken over all neighbourhoods u of s.

Let also $\rho(s, \Sigma) = \sup\{\rho_k(s, \Sigma);\ k = 1, 2, \cdots\}$ and $\rho(S, \Sigma) =\sup\{\rho(s, \Sigma);\ s\in S\}$.

Theorem 1. *If P is a projection of $B(S, \Sigma)$ onto $C(S)$, then* $\|P\| \geq 1+2\rho(S, \Sigma)$.

Proof. Let ϵ be an arbitrary positive number. Let $s_1\in S$ be such that $\rho(S, \Sigma) \leqq \rho(s_1, \Sigma) + \epsilon/8$. We choose k such that $\rho_k(s_1, \Sigma) \geqq \rho(s_1, \Sigma) - \epsilon/8$, and we take disjoint open sets $G(1, 1), \cdots, G(1, n_1)$ in Σ such that $s_1 \in \bigcap_{i=1}^{n_1} [G(1, i)]^-$ and

$$\rho_k(s_1, \Sigma) \leqq \left(1 - \frac{1}{n_1}\right) + \inf_u \min_i \sup_{t \in G(1,i) \cap u} \rho_{k-1}(t, \Sigma) + \frac{\epsilon}{16k} \cdot$$

Let $H(1, i) = G(1, i)$ for $i = 1, \cdots, n_1 - 1$, and $H(1, n_1) = S - \bigcup_{i=1}^{n_1-1} G(1, i)$.

Let $v_1 = S$, $f_1 = 1$ and $X(1, i) = \chi_{H(1,i)} f_1$, where $\chi_{H(1,i)}$ is the characteristic function of $H(1, i)$ $(i = 1, \cdots, n_1)$. Let $PX(j, i; s)$ denote the value of the function $PX(j, i)$ at the point s.

Since $\sum_{i=1}^{n_1} X(1, i) = f_1 \in C(S)$, it follows that $\sum_{i=1}^{n_1} PX(1, i; s_1) = f_1(s_1) = 1$. Hence we have for some i_1, with $1 \leqq i_1 \leqq n_1$, $PX(1, i_1; s_1) \leqq 1/n_1$. $PX(1, i_1)$ is continuous and therefore there exists a neighbourhood u_1 of s_1, $u_1 \in \Sigma$, such that for every $t \in u_1$: $|PX(1, i_1; t) - PX(1, i_1; s_1)| < \epsilon/8k$.

We proceed by induction. If $1 < j < k$ and for each r, $1 \leqq r < j$, s_r, n_r, v_r, f_r, $G(r, i)$, $H(r, i)$, $X(r, i)$ $(1 \leqq i \leqq n_r)$, i_r and u_r are defined, and satisfy the following relations:

$$u_r, v_r, G(r, i), H(r, i) \in \Sigma; \qquad f_r \in C(S);$$
$$s_r \in u_r \subset v_r \subset \bar{v}_r \subset u_{r-1} \cap G(r-1, i_{r-1});$$
$$G(r, i) \subset H(r, i) \subset G(r-1, i_{r-1}) \cap u_{r-1};$$

for $1 \leqq r < j$ (take $u_0 \cap G(0, i_0)$ as S), and also:

$$\rho_{k-r+1}(s_r, \Sigma) - \rho_{k-r}(s_{r+1}, \Sigma) \leqq \left(1 - \frac{1}{n_r}\right) + \frac{\epsilon}{8k} \qquad \text{for } 1 \leqq r < j-1,$$

$$\rho_{k-j+2}(s_{j-1}, \Sigma) \leqq \left(1 - \frac{1}{n_{j-1}}\right) + \inf_u \min_i \sup_{t \in G(j-1,i) \cap u} \rho_{k-j+1}(t, \Sigma) + \frac{\epsilon}{16k} \cdot$$

By the last inequality we can find $s_j \in u_{j-1} \cap G(j-1, i_{j-1})$ such that $\rho_{k-j+2}(s_{j-1}, \Sigma) \leqq (1 - (1/n_{j-1})) + \rho_{k-j+1}(s_j, \Sigma) + \epsilon/8k$. Let $v_j \in \Sigma$ be an open neighbourhood of s_j such that $\bar{v}_j \subset u_{j-1} \cap G(j-1, i_{j-1})$ and $f_j \in C(S)$ be a Urysohn function which is 1 on v_j, 0 outside $u_{j-1} \cap G(j-1, i_{j-1})$, and $0 \leqq f_j \leqq 1$.

There exist $G(j, 1), \cdots, G(j, n_j)$ disjoint open sets in Σ such that

$$\rho_{k-j+1}(s_j, \Sigma) \leqq \left(1 - \frac{1}{n_j}\right) + \inf_u \min_i \sup_{t \in G(j,i) \cap u} \rho_{k-j}(t, \Sigma) + \frac{\epsilon}{16k}$$

and we may assume that $G(j, i) \subset G(j-1, i_{j-1}) \cap u_{j-1}$ (by replacing G

by $G\cap G(j-1, i_{j-1})\cap u_{j-1})$. We put $H(j,i)=G(j,i)$ for $i=1,\cdots,n_j-1$, and $H(j, n_j)=u_{j-1}\cap G(j-1, i_{j-1})-\bigcup_{i=1}^{n_j-1} G(j, i)$.

Let $X(j, i)=\chi_{H(j,i)}f_j$, then $\sum_{i=1}^{n_j} X(j, i)=\chi_{G(j-1,i_{j-1})\cap u_{j-1}}f_j=f_j$, and i_j can be chosen such that $PX(j, i_j; s_j)\leqq 1/n_j$. $u_j\in\Sigma$ is an open neighbourhood of s_j contained in v_j, in which $|PX(j,i_j;t)-PX(j,i_j;s_j)|<\epsilon/8k$. All requirements are satisfied again, and we are ready to begin with $(j+1)$st step.

After $k-1$ steps we choose S_k in $u_{k-1}\cap G(k-1, i_{k-1})$ such that

$$\rho_2(s_{k-1}, \Sigma) \leqq \left(1 - \frac{1}{n_{k-1}}\right) + \rho_1(s_k, \Sigma) + \frac{\epsilon}{8k},$$

v_k, f_k as above. $G(k, 1),\cdots, G(k, n_k)$ are disjoint open sets in Σ such that $G(k, i)\subset G(k-1, i_{k-1})\cap u_{k-1}$, $s_k\in\bigcap_{i=1}^{n_k} [G(K, i)]^-$ and $\rho_1(s_k, \Sigma)\leqq(1-(1/n_k))+\epsilon/8k$. $H(k, i)$, $X(k, i)$ are defined as above and i_k is chosen such that $PX(k, i_k; s_k)\leqq 1/n_k$, $u_k\in\Sigma$ is an open neighbourhood of s_k contained in v_k in which $|PX(k, i_k; t)-PX(k, i_k; s_k)|<\epsilon/8k$.

Take an arbitrary point s_{k+1} in $u_k\cap G(k, i_k)$ and a function $f_{k+1}\in C(S)$ such that $f_{k+1}(s_{k+1})=1$, $0\leqq f_{k+1}\leqq 1$, and $f_{k+1}(S-G(k, i_k)\cap u_k)=0$.

At last, define the function

$$\begin{aligned}
F &= 1 + 2\sum_{j=2}^{k+1} f_j - 2\sum_{j=1}^{k} X(j, i_j) \\
&= f_1 + 2\sum_{j=2}^{k+1} f_j - 2\sum_{j=1}^{k} \chi_{H(j,i_j)}f_j \\
&= (1 - 2\chi_{H(1,i_1)})f_1 + 2\sum_{j=2}^{k} (1 - \chi_{H(j,i_j)})f_j + 2f_{k+1} \\
&= (1 - 2\chi_{H(1,i_1)})(1 - \chi_{H(1,i_1)}) + (1 - 2\chi_{H(2,i_2)}f_2)(\chi_{H(1,i_1)} - \chi_{H(2,i_2)}) \\
&\quad + \cdots + (1 - 2\chi_{H(k,i_k)}f_k)(\chi_{H(k-1,i_{k-1})} - \chi_{H(k,i_k)}) \\
&\quad + (2f_{k+1} - 1)\chi_{H(k,i_k)}
\end{aligned}$$

as in $H(j-1, i_{j-1})-H(j, i_j)$, f_r is 0 for $r>j$ and 1 for $r<j$.

From the last expression it is clear that $-1\leqq F\leqq 1$. Since the f_i are continuous, $PF=1+2\sum_{j=2}^{k+1} f_j-2\sum_{j=1}^{k} PX(j, i_j)$; hence

$$PF(s_{k+1}) = 1 + 2k - 2\sum_{j=1}^{k} PX(j, i_j; s_{k+1}).$$

But $s_{k+1}\in\bigcap_{j=1}^{k} u_j$, hence

$$\begin{aligned}
\|P\| &\geqq PF(s_{k+1}) \geqq 1 + 2k - 2\left(\sum_{j=1}^{k}\frac{1}{n_j} - \frac{\epsilon}{8k}\right) \\
&= 1 + 2\sum_{j=1}^{k}\left(1 - \frac{1}{n_j}\right) - \frac{\epsilon}{4} \\
&\geqq 1 + 2\left\{\sum_{j=2}^{k}\left[\rho_{k-j+2}(s_{j-1}, \Sigma) - \rho_{k-j+1}(s_j, \Sigma) - \frac{\epsilon}{8k}\right]\right. \\
&\qquad\qquad\left. + \rho_1(s_k, \Sigma) - \frac{\epsilon}{8k}\right\} - \frac{\epsilon}{4} \\
&= 1 + 2\rho_k(s_1, \Sigma) - \frac{\epsilon}{2} \geqq 1 + 2\rho(s_1, \Sigma) - \frac{5\epsilon}{8} \geqq 1 + 2\rho(S, \Sigma) - \epsilon.
\end{aligned}$$

This completes the proof of Theorem 1.

Let D_Σ denote the set of all boundary points of the closures of open sets in Σ, i.e., $D_\Sigma = \{s \in S;\ s \in \operatorname{bd} \overline{G},\ \Sigma \ni G \text{ open}\}$.

Corollary 1. *If there exists a bounded projection from $B(S, \Sigma)$ onto $C(S)$, then D_Σ is nowhere dense in S.*

Proof. Suppose, on the contrary, that D_Σ is dense in an open U. If $s \in \operatorname{bd} \overline{G}$ is an arbitrary point in D_Σ, then $s \in \overline{G} \cap [S - \overline{G}]^-$, hence $\rho_1(s, \Sigma) \geqq \frac{1}{2}$. By induction it follows that for every $s \in U \cap D_\Sigma$ we have $\rho_k(s, \Sigma) \geqq \frac{1}{2}k$, hence $\rho(S, \Sigma) = \infty$, which contradicts the existence of a bounded projection.

3. P_λ **spaces of the** $C(S)$ **type.** Now we apply the results of the previous section to P_λ spaces of the $C(S)$ type. If Σ is the field of all subsets of S, we shall write $\rho(S)$ for $\rho(S, \Sigma)$. $B(S, \Sigma)$ in this case is simply $M(S)$. In this case Theorem 1 becomes:

Theorem 1′. *If $C(S)$ is a P_λ space, then $\lambda \geqq 1 + 2\rho(S)$.*

From Corollary 1 we get:

Theorem 2. *If $C(S)$ is a P_λ space, then S contains a dense open extremally disconnected subset.*

Proof. Let $D = \{s \in S: s \in \operatorname{bd} \overline{G},\ G \text{ open}\}$. By Corollary 1, $\Omega = S - \overline{D}$ is a dense subset of S which is open by definition. If G is an open subset of Ω, G is open in S too. The closure of G in Ω is contained in $\overline{G}$, hence the boundary of this closure is contained in $\operatorname{bd} \overline{G}$ and therefore in D, and must be empty. This proves that Ω is also extremally disconnected. q.e.d.

Corollary 2. *If X is an infinite discrete space, then $C(\beta X - X)$ is not a P_λ space. (βS denotes the Stone-Čech compactification of S.)*

PROOF. Let N be an infinite countable subset of X. $C(\beta N-N)$ is a direct factor of $C(\beta X-X)$, hence it is enough to prove that $C(\beta N-N)$ is not a P_λ space. If it were a P_λ space, then $\beta N-N$ would contain an open dense extremally disconnected subset, and therefore it would contain an open and closed nonvoid extremally disconnected subset of $\beta N-N$. But by a theorem of Rudin [9] an open and closed nonvoid subset of $\beta N-N$ is homeomorphic to $\beta N-N$, which is not extremally disconnected. The contradiction reached proves our assertion.

REMARKS. a. $C(\beta X-X)$ is a continuous image of the P_1 space $C(\beta X)$.

b. Though the finiteness of $\rho(S)$ is a necessary condition for $C(S)$ to be a P_λ space, it is not sufficient. A simple counterexample is the space $c=C(N^*)$ (where N^* denotes the one-point compactification of the discrete sequence N). Since N^* contains a convergent sequence, $C(N^*)$ cannot be a P_λ space, although $\rho(N^*)=1$. A more interesting example is the following: Let X' and X'' be two homeomorphic discrete sets, and let S be the space obtained from $\beta X'\cup\beta X''$ by the identification of the naturally corresponding points of $\beta X'-X'$ and $\beta X''-X''$. $\rho(S)=\frac{1}{2}$, but $C(S)$ contains c_0 as a direct factor, hence is not a P_λ space [6].

On the other hand, if we identify only two corresponding points: $s_1\in\beta X'-X'$ and $s_2\in\beta X''-X''$, we have a P_2 space with $\rho(S)=\frac{1}{2}$.

c. Starting with an infinite, extremally disconnected, compact space, and using the two procedures:

1. "Binding" a finite number of copies of a space in one point.

2. Taking the Stone-Čech compactification of the union of a family of spaces.

We can construct S_ρ for every possible ρ (i.e., belonging to the closure of $\{\sum_{i=1}^{k}(1-1/n_i);\ k,\ n_i \text{ natural}\}$ in the real line) such that $\rho(S_\rho)=\rho$ and $C(S_\rho)$ is a $P_{1+2\rho}$ space (hence $1+2\rho$ is exact).

4. P'_λ spaces of the $C(S)$ type.

THEOREM 3. *The following statements are equivalent*:

1. *$C(S)$ is a P'_λ space for some finite λ.*

2. *S is homeomorphic to the space of ordinals $\{\eta;\ \eta\leqq\zeta\}$, with the order topology, for some $\zeta<\omega^\omega$.*

3. *$C(S)$ is isomorphic to c_0.*

PROOF. The implications $3\Rightarrow 1$ and $2\Rightarrow 3$ are simple. To prove $1\Rightarrow 2$ we shall show first that $D=\{s\in S;\ s\in \text{bd}\,\overline{G},\ G \text{ open}\}$ is nowhere dense in S. Suppose that $\overline{D}$ contains an open nonvoid U. Since $C(S)$ is separable, S is metrizable and we can find a countable family of open sets $\{G_i;\ i=1,2,\cdots\}$ such that $U\subset[\bigcup_{i=1}^{\infty}\text{bd}\,\overline{G}_i]^-$.

Let H be a countable open basis in S which contains all the G_i, and let Σ be the field generated by H and the operations: closure, complementation, and finite union. Σ is countable, and $B(S, \Sigma)$ is a separable Banach space containing $C(S)$, therefore there exists a bounded projection of $B(S, \Sigma)$ onto $C(S)$. By Corollary 1, D_Σ is nowhere dense in S, but this leads to contradiction since we assumed $U \subset [\bigcup_{i=1}^\infty \text{bd}\, \overline{G}_i]^- \subset D_\Sigma$.

$S-\overline{D}$ is an open, dense, extremally disconnected subset of S, and contains an open and closed, nonvoid, extremally disconnected subset. Since S is metrizable, such a subset can be only finite. Hence S contains an isolated point and cannot be perfect.

Next we note that if $C(S)$ is a P'_λ space and A is a closed subset of S, then $C(A)$ is also a P'_λ space. Indeed, by a theorem of Dugundji [2], $C(A)$ is isometric to a subspace of $C(S)$ onto which there is a projection with norm 1.

This implies that if $C(S)$ is a P'_λ space for some finite λ, then no nonvoid subset of S is perfect. By a theorem of Sierpiński (Pełczyński and Semadeni [8]), S is in this case homeomorphic to the space of ordinals $\{\eta;\ \eta \leqq \zeta\}$ (with the order topology) for some $\zeta < \omega_1$.

To prove that $\zeta < \omega^\omega$ we note that if $\omega_1 > \zeta \geqq \omega^k$, then $\rho_k(\omega^k, \Sigma) = k$, where Σ is the countable field of subsets in $\{\eta;\ \eta \leqq \zeta\}$ generated by the (countable) basis of order intervals. Applying again Theorem 1 we conclude our proof.

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