SOME RESULTS ON THE ASYMPTOTIC COMPLETION OF AN IDEAL

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1. **Introduction.** Let $\mathfrak{o}$ be a commutative ring with identity. A *semi-prime* operation on $\mathfrak{o}$ is a mapping $\mathfrak{a}\rightarrow\mathfrak{a}_p$ of $\mathfrak{o}$-ideals into $\mathfrak{o}$-ideals which satisfies all the conditions of a prime operation in the sense of Krull [1; 2], except that one does not require that $(x\mathfrak{a})_p=x(\mathfrak{a})_p$ for all $x\in\mathfrak{o}$ and all ideals $\mathfrak{a}$. Specifically, a semi-prime operation satisfies the conditions

$$(1)\qquad \begin{aligned} &\text{(i)} && \mathfrak{a}\subseteq\mathfrak{a}_p,\\ &\text{(ii)} && \mathfrak{a}\subseteq\mathfrak{b} \text{ implies that } \mathfrak{a}_p\subseteq\mathfrak{b}_p,\\ &\text{(iii)} && \mathfrak{a}_{pp}=\mathfrak{a}_p,\\ &\text{(iv)} && \mathfrak{a}_p\mathfrak{b}_p\subseteq(\mathfrak{a}\mathfrak{b})_p. \end{aligned}$$

Formal consequences [5] of the foregoing definition are (v) $\mathfrak{o}_p=\mathfrak{o}$, (vi) $(\mathfrak{a}_p\mathfrak{b}_p)_p=(\mathfrak{a}\mathfrak{b})_p$, (vii) $\mathfrak{a}_p\subseteq\mathfrak{b}_p$ and $\mathfrak{c}$ arbitrary imply that $(\mathfrak{a}\mathfrak{c})_p\subseteq(\mathfrak{b}\mathfrak{c})_p$, (viii) $(\sum_\alpha\mathfrak{a}_\alpha)_p=(\sum_\alpha(\mathfrak{a}_\alpha)_p)_p$, and (ix) $\bigcap_\alpha(\mathfrak{a}_\alpha)_p=(\bigcap_\alpha(\mathfrak{a}_\alpha)_p)_p$, where (viii) and (ix) are arbitrary sums and intersections, respectively.

The identity operation $\mathfrak{a}\rightarrow\mathfrak{a}$ and the radical operation $\mathfrak{a}\rightarrow\operatorname{Rad}\mathfrak{a}$ are trivial examples of semi-prime operations. If $\mathfrak{o}$ is an integrally closed domain and if $\mathfrak{a}_a$ is the integral completion of $\mathfrak{a}$ (see [1; 2]), it is well known that the a-operation $\mathfrak{a}\rightarrow\mathfrak{a}_a$ is a prime operation. If $\mathfrak{o}$ is an arbitrary integral domain the a-operation is still a semi-prime operation. In §6 it is shown by a variation of the classical argument that the a-operation is always a semi-prime operation, even if $\mathfrak{o}$ is not an integral domain.

If $\mathfrak{a}$ is an $\mathfrak{o}$-ideal let $v_\mathfrak{a}(x)=n$ in case $x\in\mathfrak{a}^n$ and $x\notin\mathfrak{a}^{n+1}$ and let $v_\mathfrak{a}(x)=\infty$ in case $x\in\mathfrak{a}^m$ for all m. Rees proved [4] that $\lim_{n\to\infty}v_\mathfrak{a}(x^n)/n$ exists for all $x\in\mathfrak{o}$ and that the function $\bar{v}_\mathfrak{a}$ defined by $\bar{v}_\mathfrak{a}(x)=\lim_{n\to\infty}v_\mathfrak{a}(x^n)/n$ is a homogeneous pseudo-valuation on $\mathfrak{o}$. The *asymptotic completion* of the ideal $\mathfrak{a}$ is defined to be

$$(2)\qquad \mathfrak{a}_s=\{x\in\mathfrak{o}\colon \bar{v}_\mathfrak{a}(x)\geqq 1\}.$$

In case $\mathfrak{o}$ is noetherian, this definition of asymptotic completion of an

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ideal is equivalent to one given by Samuel [6], and furthermore, the s-operation $\mathfrak{a} \to \mathfrak{a}_s$ is precisely the a-operation [3; 4].

In this paper the following result is proven for the arbitrary (not necessarily noetherian) case:

THEOREM. *The s-operation* $\mathfrak{a} \to \mathfrak{a}_s$ *is always a semi-prime operation which satisfies the cancellation law*

$$(3) \qquad (\mathfrak{a}\mathfrak{c})_s \subseteq (\mathfrak{b}\mathfrak{c})_s \text{ and } \mathfrak{a} \subseteq \operatorname{Rad} \mathfrak{c} \text{ together imply that } \mathfrak{a}_s \subseteq \mathfrak{b}_s.$$

Moreover, $\mathfrak{a}_s = \mathfrak{b}_s$ *if and only if* $\bar{v}_{\mathfrak{a}} = \bar{v}_{\mathfrak{b}}$.

A consequence of the cancellation law (3) is that, if $(\mathfrak{a}\mathfrak{c})_s = (\mathfrak{b}\mathfrak{c})_s$ and $\mathfrak{a} + \mathfrak{b} \subseteq \operatorname{Rad} \mathfrak{c}$, then $\mathfrak{a}_s = \mathfrak{b}_s$.

However, the s-operation is in general not the same as the a-operation, although it is always true that $\mathfrak{a}_a \subseteq \mathfrak{a}_s$. In particular, $(x\mathfrak{o})_s$ need not be $x\mathfrak{o}$, even if $\mathfrak{o}$ is an integrally closed domain. In §5 a characterization is given of those integral domains in which all principal ideals are asymptotically complete.[2]

2. **Preliminary results.** By definition of a homogeneous pseudo-valuation, $\bar{v}_{\mathfrak{a}}$ satisfies the conditions

$$(4) \qquad \begin{aligned} &\text{(i)} \quad \bar{v}_{\mathfrak{a}}(x \pm y) \geqq \min(\bar{v}_{\mathfrak{a}}(x), \bar{v}_{\mathfrak{a}}(y)), \\ &\text{(ii)} \quad \bar{v}_{\mathfrak{a}}(xy) \geqq \bar{v}_{\mathfrak{a}}(x) + \bar{v}_{\mathfrak{a}}(y), \\ &\text{(iii)} \quad \bar{v}_{\mathfrak{a}}(x^n) = n\bar{v}_{\mathfrak{a}}(x) \text{ for all positive integers } n. \end{aligned}$$

It follows that $\mathfrak{a}_s$ as defined by (2) is an ideal. Also, one easily sees that

$$(5) \qquad \begin{aligned} &\text{(i)} \quad \mathfrak{a} \subseteq \mathfrak{b} \text{ implies that } \bar{v}_{\mathfrak{a}} \leqq \bar{v}_{\mathfrak{b}}, \\ &\text{(ii)} \quad \bar{v}_{\mathfrak{a}} \leqq \bar{v}_{\mathfrak{b}} \text{ implies that } \mathfrak{a}_s \subseteq \mathfrak{b}_s. \end{aligned}$$

LEMMA 1. *A necessary and sufficient condition that* $\bar{v}_{\mathfrak{a}}(x) \geqq \alpha > 0$ *is that for every rational number* $0 < p/q < \alpha$ *there exists a positive integer* k *such that* $x^{qk} \in \mathfrak{a}^{pk}$. *In particular, if* $x \in \mathfrak{a}_s$, *then for every positive integer* n, *there exists a positive integer* k *such that* $x^{(n+1)k} \in \mathfrak{a}^{nk}$.

PROOF. $\bar{v}_{\mathfrak{a}}(x) = \sup\{v_{\mathfrak{a}}(x^n)/n : n \text{ a positive integer}\}$.

PROPOSITION 2. *If* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are ideals and if* $x \in \mathfrak{o}$, *then*

$$\text{(i)} \quad \frac{1}{\bar{v}_{\mathfrak{a}\mathfrak{b}}(x)} \leqq \frac{1}{\bar{v}_{\mathfrak{a}}(x)} + \frac{1}{\bar{v}_{\mathfrak{b}}(x)},$$

$$\text{(ii)} \quad \bar{v}_{\mathfrak{a}^n}(x) = \frac{1}{n}\bar{v}_{\mathfrak{a}}(x), \qquad n \text{ a positive integer.}$$

[2] An ideal $\mathfrak{a}$ is *asymptotically complete,* or *s-complete,* in case $\mathfrak{a}_s = \mathfrak{a}$.

PROOF. Assume $x \in \operatorname{Rad} \mathfrak{a}\mathfrak{b}$, for otherwise (i) reduces to $\infty \leqq \infty$. This implies that $\bar{v}_{\mathfrak{a}}(x) = \alpha > 0$ and $\bar{v}_{\mathfrak{b}}(x) = \beta > 0$. If $0 < n/m < \alpha$ and $0 < p/q < \beta$ then by Lemma 1 there exist j and k such that $x^{mj} \in \mathfrak{a}^{nj}$ and $x^{qk} \in \mathfrak{b}^{pk}$. It follows that $x^{(mp+nq)jk} \in (\mathfrak{a}\mathfrak{b})^{npjk}$. Hence $\bar{v}_{\mathfrak{a}\mathfrak{b}}(x) \geqq np/(mp+nq)$. To obtain (i) let $n/m \to \alpha$ and $p/q \to \beta$.

If $0 < p/q < \bar{v}_{\mathfrak{a}}^{n}(x)$ then for some k, $x^{qk} \in \mathfrak{a}^{npk}$. This implies that $\bar{v}_{\mathfrak{a}}(x) \geqq n(p/q)$, which in turn implies that $\bar{v}_{\mathfrak{a}}(x) \geqq n\bar{v}_{\mathfrak{a}^n}(x)$. The opposite inequality required for (ii) follows from (i).

PROPOSITION 3. *If* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are ideals, then for all* $x \in \mathfrak{o}$, $\lim_{n\to\infty} n\bar{v}_{\mathfrak{a}^n\mathfrak{b}}(x)$ *exists. Moreover, the value of the limit is* $\bar{v}_{\mathfrak{a}}(x)$ *if* $x \in \operatorname{Rad} \mathfrak{b}$, *and is zero otherwise. In particular, if* $\mathfrak{a} \subseteq \operatorname{Rad} \mathfrak{b}$, *then for all* $x \in \mathfrak{o}$,

$$\lim_{n\to\infty} n\bar{v}_{\mathfrak{a}^n\mathfrak{b}}(x) = \bar{v}_{\mathfrak{a}}(x).$$

PROOF. Since $\mathfrak{a}^n\mathfrak{b} \subseteq \mathfrak{a}^n$ it follows by (5) and Proposition 2 that $n\bar{v}_{\mathfrak{a}^n\mathfrak{b}} \leqq n\bar{v}_{\mathfrak{a}^n} = \bar{v}_{\mathfrak{a}}$. On the other hand, if $x \in \operatorname{Rad} \mathfrak{b}$, it is seen from Proposition 2 that

$$\limsup_{n\to\infty} \frac{1}{n\bar{v}_{\mathfrak{a}^n\mathfrak{b}}(x)} \leqq \limsup_{n\to\infty} \left[\frac{1}{\bar{v}_{\mathfrak{a}}(x)} + \frac{1}{n\bar{v}_{\mathfrak{b}}(x)}\right] = \frac{1}{\bar{v}_{\mathfrak{a}}(x)} \cdot$$

Thus $\bar{v}_{\mathfrak{a}}(x) \leqq \liminf_{n\to\infty} n\bar{v}_{\mathfrak{a}^n\mathfrak{b}}(x) \leqq \limsup_{n\to\infty} n\bar{v}_{\mathfrak{a}^n\mathfrak{b}}(x) \leqq \bar{v}_{\mathfrak{a}}(x)$.

3. **Proof of Theorem.** From definition (2) and conditions (5) it follows that the s-operation satisfies (i) and (ii) of (1). It now will be shown that (iv) of (1) holds, that is

$$\mathfrak{a}_s\mathfrak{b}_s \subseteq (\mathfrak{a}\mathfrak{b})_s. \tag{6}$$

For, suppose that $x \in \mathfrak{a}_s$ and $y \in \mathfrak{b}_s$. If n is a positive integer there exists by Lemma 1 a positive integer k such that $x^{(n+1)k} \in \mathfrak{a}^{nk}$ and $y^{(n+1)k} \in \mathfrak{b}^{nk}$. Hence $(xy)^{(n+1)k} \in (\mathfrak{a}\mathfrak{b})^{nk}$. This implies that $\bar{v}_{\mathfrak{a}\mathfrak{b}}(xy) \geqq 1$, which establishes (6). A consequence of (6) is that by induction, for every positive integer n,

$$(\mathfrak{a}_s)^n \subseteq (\mathfrak{a}^n)_s.$$

To establish that the s-operation satisfies condition (iii) of (1) it will be sufficient to show that for all $x \in \mathfrak{o}$

$$\bar{v}_{\mathfrak{a}}(x) = \bar{v}_{\mathfrak{a}_s}(x). \tag{7}$$

Suppose $0 < r/t < \bar{v}_{\mathfrak{a}_s}(x)$. It follows that for suitable k,

$$x^{tk} \in (\mathfrak{a}_s)^{rk} \subseteq (\mathfrak{a}^{rk})_s.$$

Hence $\bar{v}_{\mathfrak{a}}(x) = rk\bar{v}_{\mathfrak{a}^{rk}}(x) = (r/t)\bar{v}_{\mathfrak{a}^{rk}}(x^{tk}) \geqq r/t$. It is thus seen that $\bar{v}_{\mathfrak{a}}(x)$

$\geqq \bar{v}_{\mathfrak{a}_s}(x)$. The opposite inequality required for (7) follows from the fact that $\mathfrak{a} \subseteq \mathfrak{a}_s$.

Now it will be shown that the cancellation law (3) holds. From $(\mathfrak{a}\mathfrak{c})_s \subseteq (\mathfrak{b}\mathfrak{c})_s$ it follows that

$$(\mathfrak{a}^2\mathfrak{c})_s \subseteq (\mathfrak{a}\mathfrak{b}\mathfrak{c})_s \subseteq (\mathfrak{b}^2\mathfrak{c})_s,$$

and by induction it is seen that

$$(\mathfrak{a}^n\mathfrak{c})_s \subseteq (\mathfrak{b}^n\mathfrak{c})_s.$$

Hence for all n, $(\mathfrak{a}^n\mathfrak{c})_s \subseteq (\mathfrak{b}^n)_s$, which implies that

$$\bar{v}_{\mathfrak{a}^n\mathfrak{c}} \leqq \bar{v}_{\mathfrak{b}^n} = \frac{1}{n}\bar{v}_{\mathfrak{b}}.$$

Since $\mathfrak{a} \subseteq \operatorname{Rad} \mathfrak{c}$, Proposition 3 yields that

$$\bar{v}_{\mathfrak{a}} = \lim_{n \to \infty} n\bar{v}_{\mathfrak{a}^n\mathfrak{c}} \leqq \bar{v}_{\mathfrak{b}}.$$

Hence $\mathfrak{a}_s \subseteq \mathfrak{b}_s$.

The last statement of the Theorem follows from (7).

4. **Remarks.** A principal ideal need not be s-complete, even if $\mathfrak{o}$ is an integrally closed domain. For example, let $K[x, y]$ be the polynomial ring in two indeterminates over a field K. Consider the valuation v defined on $K[x, y]$ as follows: If $f(x, y) = \sum a_{ij}x^iy^j$, then $v(f) = \min\{(i,j) : a_{ij} \neq 0\}$ where the pairs (i, j) are ordered lexicographically. The associated valuation ring R_v is a rank 2, discrete valuation ring. The maximal ideal of R_v is $M_v = yR_v$. The other nonzero prime ideal is $P = (xy^{-m} : m = 1, 2, \cdots)R_v$. Obviously xR_v is a proper subideal of P. However, $(xR_v)_s = P$. To show this it will be sufficient to show that $xy^{-m} \in (xR_v)_s$ for all positive m. Clearly for any $n \geqq 1$, $(xy^{-m})^{n+1} = x^n(xy^{-m(n+1)})$, and hence $(xy^{-m})^{n+1} \in x^nR_v$. This implies that $xy^{-m} \in (xR_v)_s$.

The radical restriction in the cancellation law (3) is essential. Consider the valuation ring R_v in the foregoing example. It is easy to verify that $M_vP = P = M_v^2P$, and that $(M_v^n)_s = M_v^n$ for all n. Hence $(M_vP)_s \subseteq (M_v^2P)_s$ but $(M_v)_s \not\subseteq (M_v^2)_s$. However, it is noted that $M_v \not\subseteq \operatorname{Rad} P = P$.

5. **A result for integral domains.** In §4 it was pointed out that in general a principal ideal was not s-complete. The following proposition characterizes those integral domains in which all principal ideals are s-complete. Let K be the field of quotients of an integral domain $\mathfrak{o}$. An element $\alpha \in K$ is *almost integral over* $\mathfrak{o}$ in case there exists a non-

zero $y \in \mathfrak{o}$ such that $y\alpha^n \in \mathfrak{o}$ for all n. The set of all elements in K which are almost integral over $\mathfrak{o}$ forms an overring $\hat{\mathfrak{o}}$ of $\mathfrak{o}$. If $\hat{\mathfrak{o}} = \mathfrak{o}$ then $\mathfrak{o}$ is said to be *completely integrally closed*. See [1].

PROPOSITION 4. *All principal ideals in an integral domain* $\mathfrak{o}$ *are s-complete if and only if* $\mathfrak{o}$ *is completely integrally closed.*

PROOF. Assume $\mathfrak{o}$ is completely integrally closed. If $x, y \neq 0$ and $y \in (x\mathfrak{o})_s$, then for each positive integer n there exists a positive integer k_n such that $y^{(n+1)k_n} \in x^{nk_n}\mathfrak{o}$. It follows that $(y^{n+1}/x^n)^{k_n} \in \mathfrak{o}$, and hence y^{n+1}/x^n is integral over $\mathfrak{o}$ for each n. But $\mathfrak{o}$ completely integrally closed implies that $\mathfrak{o}$ is integrally closed. Thus for all n, $y(y/x)^n = y^{n+1}/x^n \in \mathfrak{o}$. From the hypothesis it follows that $y/x \in \mathfrak{o}$, and hence $y \in x\mathfrak{o}$.

On the other hand assume that every principal ideal is s-closed. If $y/x \in K$ is almost integral over $\mathfrak{o}$, then for some nonzero $z \in \mathfrak{o}$, $z(y/x)^n \in \mathfrak{o}$ for all n. Hence $zy^n \in x^n\mathfrak{o}$ for all n. It follows that $(zy)^{n+1} = yz^n(zy^n) \in yz^nx^n\mathfrak{o} \subseteq (zx)^n\mathfrak{o}$. This implies that $zy \in (zx\mathfrak{o})_s = zx\mathfrak{o}$. Hence $y/x \in \mathfrak{o}$.

6. **The a-operation.** An element $x \in \mathfrak{o}$ is *integral over* an ideal $\mathfrak{a}$ in case for some n there exist $a_i \in \mathfrak{a}^i$, $i = 1, \cdots, n$, such that

$$x^n + a_1x^{n-1} + \cdots + a_n = 0.$$

The set of all elements integral over $\mathfrak{a}$ is denoted by $\mathfrak{a}_a$. In case $\mathfrak{o}$ is an integral domain it is well known [2] that

(8) $x \in \mathfrak{a}_a$ iff there exists a finitely generated nonzero ideal $\mathfrak{b}$ such that $x\mathfrak{b} \subseteq \mathfrak{a}\mathfrak{b}$.

From this it easily follows that $\mathfrak{a}_a$ is an ideal, called the integral completion of $\mathfrak{a}$, and that the a-operation $\mathfrak{a} \to \mathfrak{a}_a$ is a semi-prime operation for which the following cancellation law holds:

If both $(\mathfrak{a}\mathfrak{c})_a \subseteq (\mathfrak{b}\mathfrak{c})_a$ *and* $\mathfrak{c}_a = \mathfrak{c}'_a$ *for some finitely generated nonzero ideal* $\mathfrak{c}'$, *then* $\mathfrak{a}_a \subseteq \mathfrak{b}_a$.

In case $\mathfrak{o}$ is not necessarily an integral domain a variation of the classical argument shows the following modification of (8) to hold:

(9)
(i) $x \in \mathfrak{a}_a$ iff there exists a finitely generated ideal $\mathfrak{b}$ such that both $x \in \operatorname{Rad} \mathfrak{b}$ and $x\mathfrak{b} \subseteq \mathfrak{a}\mathfrak{b}$.
(ii) The ideal $\mathfrak{b}$ above can be chosen so that its radical contains any finite number of elements in the radical of $\mathfrak{a}$.

From (9) it still follows that $\mathfrak{a}_a$ is an ideal and that the a-operation is a semi-prime operation. Moreover, the following cancellation law, shown by Nagata [3] to hold in the noetherian case, holds in general:

If $(\mathfrak{a}\mathfrak{c})_a \subseteq (\mathfrak{b}\mathfrak{c})_a$, *if* $\mathfrak{c}_a = \mathfrak{c}'_a$ *for some finitely generated ideal* $\mathfrak{c}'$, *and if* $\mathfrak{a}$ *is contained in every minimal prime ideal of* (0) *in which* $\mathfrak{c}$ *is contained, then* $\mathfrak{a}_a \subseteq \mathfrak{b}_a$.

Finally, it will be noted that from (9(i)) and (3) it follows that $\mathfrak{a}_a \subseteq \mathfrak{a}_s$ always. This result also follows directly from the definition of $\mathfrak{a}_a$. See [4; 5].

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