NOTE ON M-GROUPOIDS

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In a recent paper of Tamura, Merkel, and Latimer [2], the following question was raised:

Suppose S is a groupoid (cf. [1]) which satisfies:

(1) There is at least one left identity in S.

(2) If y or z is a left identity of S, then $x(yz)=(xy)z$ for all $x \in S$.

(3) For all a, $b \in S$ there exists $x \in S$ such that $ax=b$.

Then does S satisfy:

(3′) For any $x \in S$ there is a unique left identity e (which may depend on x) such that $xe=x$?

A groupoid which satisfies (1), (2), and (3′) is defined in [2] to be an M-groupoid. It is the purpose of this note to present an example of a groupoid satisfying (1), (2), and (3), which is not an M-groupoid. It will be shown, however, that every finite groupoid satisfying (1), (2), and (3) is an M-groupoid.

Let A be a denumerable set. For simplicity, denote its elements by $1, 2, 3, \cdots$. Let $\cdot$ be a binary operation on A which satisfies the following Cayley table:

	1	2	3	4	·	·	·	m	·	·	·
1	1	2	3	4	·	·	·	m	·	·	·
2	3	2	3	a_{24}	·	·	·	a_{2m}	·	·	·
3	3	2	3	a_{34}	·	·	·	a_{3m}	·	·	·
4	4	2	3	a_{44}	·	·	·	a_{4m}	·	·	·
·	·	·	·	·				·			
·	·	·	·	·				·			
·	·	·	·	·				·			
n	n	2	3	a_{n4}	·	·	·	a_{nm}	·	·	·
·	·	·	·	·				·			
·	·	·	·	·				·			
·	·	·	·	·				·			,

where the a_{ij} are arbitrary positive integers subject only to the restrictions:

Received by the editors April 6, 1963.

(a) $a_{ij} \neq 2$ for all $i>1, j>3$.
(b) $a_{2j}=a_{3j}$ for all $j>3$.
(c) For each $i>1$, $\{a_{ij}: j>3\}=A$.

That is, the operation $\cdot$ is to satisfy the following conditions:

(i) $1 \cdot x = x$ for all $x \in A$.

(ii) $x \cdot 1 = \begin{cases} x \text{ if } x \neq 2, \\ 3 \text{ if } x = 2. \end{cases}$

(iii) $x \cdot y = 2$ if and only if $y=2$.

(iv) $x \cdot 3 = 3$ for all $x \in A$.

(v) $2 \cdot x = 3 \cdot x$ for all $x \in A$.

(vi) For all $x \in A$, $\{x \cdot y : y \in A\} = A$.

(In particular, if we let $a_{i4}=1$ for all $i>1$, and $a_{ij}=j-1$ for all $i>1$, $j>4$, then the binary operation defined by the Cayley table above satisfies (i)–(vi). Thus these conditions are consistent.)

Then $S=\langle A, \cdot \rangle$ is a groupoid which satisfies conditions (1) and (3), by (i) and (vi). To see that S satisfies (2), consider the two cases:

(α) Suppose $y=1$ in (2). If $x \neq 2$ then $(x \cdot 1) \cdot z = x \cdot z = x \cdot (1 \cdot z)$, by (ii) and (i). If $x=2$ then $(x \cdot 1) \cdot z = (2 \cdot 1) \cdot z = 3 \cdot z = 2 \cdot z = 2 \cdot (1 \cdot z)$ $= x \cdot (1 \cdot z)$, by (ii), (v), and (i). Thus we have $(x \cdot 1) \cdot z = x \cdot (1 \cdot z)$ for all $x, z \in S$.

(β) Suppose $z=1$ in (2). If $y \neq 2$ then $x \cdot y \neq 2$ and $(x \cdot y) \cdot 1 = x \cdot y$ $= x \cdot (y \cdot 1)$, by (iii) and (ii). If $y=2$ then $x \cdot y = 2$ and $(x \cdot y) \cdot 1 = 2 \cdot 1 = 3$ $= x \cdot 3 = x \cdot (2 \cdot 1) = x \cdot (y \cdot 1)$, by (iii), (ii), and (iv). Thus we have $(x \cdot y) \cdot 1 = x \cdot (y \cdot 1)$ for all $x, y \in S$.

Since $x \neq 1$ implies $x \cdot 1 \neq 1$ by (ii), then 1 is the only left identity of S. Hence S satisfies (2), by (α) and (β).

However, S is not an M-groupoid because (3′) is not satisfied for $x=2$.

Suppose (3) is replaced by the stronger condition:

(3″) For all $a, b \in S$ there exists a unique $x \in S$ such that $ax=b$. Then it is easy to prove the following

THEOREM. *If a groupoid S satisfies* (1), (2) *and* (3″) *then S is an M-groupoid.*

PROOF. Let $a \in S$. By (3″) there is a unique $c \in S$ such that $ac=a$. To show that S is an M-groupoid it suffices, by (3′), to show that c is a left identity of S. By (1) and (2), there exists a left identity e of S such that $a(ce)=(ac)e=ae$, and hence $ce=e$, by (3″). Therefore, for any $x \in S$, we have $cx=c(ex)=(ce)x=ex=x$, and the proof is completed.

Since every finite groupoid satisfying (3) also satisfies (3″), we have the immediate

COROLLARY. *If S is a finite groupoid which satisfies* (1), (2) *and* (3), *then S is an M-groupoid.*

REFERENCES

1. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol. 1, Math. Surveys No. 7, Amer. Math. Soc., Providence, R. I., 1961.

2. T. Tamura, R. B. Merkel and J. F. Latimer, *The direct product of right singular semigroups and certain groupoids*, Proc. Amer. Math. Soc. **14** (1963), 118–123.

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THE INDEX PROBLEM FOR INFINITE SYMMETRIC GROUPS

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Let M be an infinite set with cardinal X, $S(X, Y)=\{\sigma\colon \sigma$ is a permutation on M such that $|\operatorname{spt}\sigma|<Y\}$, where $\operatorname{spt}\sigma=\{m\in M\colon \sigma(m)\neq m\}$. If X is a cardinal, denote its successor by X^*. Onofri [2] proved that $S(d, d^*)$ has no proper subgroups of finite index and $S(d, d)$ has precisely one, the alternating group $A(d)$. These results have been extended by Higman [1] and Scott [3]. Higman has shown that $S(X, d)$ has only one proper subgroup of index less than X, the alternating group $A(X)$, and $A(X)$ has no proper subgroups of index less than X. If Z is a cardinal such that $Z^Z<X$, Scott proved that $S(X, Y)$ for $Y>d$ has no subgroups of index less than or equal Z.

In this paper, the following generalization of these results is proven.

THEOREM. *If $d<Y\leqq X^*$, $S(X, Y)$ has no proper subgroups of index less than X.*

LEMMA 1. *If $d\leqq Z<Y$, $[S(X, Y)\colon H]<X$, then H contains an element σ such that $|\operatorname{spt}\sigma|=Z$ and $|M\backslash\operatorname{spt}\sigma|=X$.*

PROOF. $H\cap S(X, Z^*)$ has index less than X in $S(X, Z^*)$ for any $Z<Y$. Since the index of $S(X, Z)$ in $S(X, Z^*)$ is greater than or equal X, there is $\sigma\in H$ such that $|\operatorname{spt}\sigma|=Z$. If $Z<X$, then $|M\backslash\operatorname{spt}\sigma|=X$. If $Z=X$, there are disjoint sets M_1 and M_2 such that $M=M_1\cup M_2$ and $|M_1|=|M_2|=X$. Let $G=\{\sigma\colon \operatorname{spt}\sigma\subset M_1\}$. Then $H\cap G$ has index less than X in G, hence there is $\sigma\in H\cap G$ such that $|\operatorname{spt}\sigma|=X$ and since $\operatorname{spt}\sigma\subset M_1$, $|M\backslash\operatorname{spt}\sigma|=X$.

Presented to the Society, April 27, 1963; received by the editors March 12, 1963.