INVERSION INTEGRALS INVOLVING JACOBI'S POLYNOMIALS

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Recently, some inversion integrals for integral equations involving either a Chebyshev or a Legendre or a Gegenbauer polynomial in the kernel have been given [1], [2], [5]. Here, by using Mellin transformation, two inversion integrals for integral equations having Jacobi's polynomial in the kernel have been obtained. We write, for k and n integers with $0 < k < n$ and for $-1 < \beta < 1$,

$$F_n^{(k,\beta)}(x) = \frac{(n)!}{2^{k-1/2}\Gamma(k-\beta+n+1)} x(x^2-1)^{k-\beta} P_n^{(k-\beta,\beta)}(2x^2-1),$$

$$G_n^{(k,\beta)}(x) = \frac{\Gamma(n-k+1)}{2^{k-1/2}\Gamma(n+\beta)} (1-x^2)^{k+\beta-1} P_{n-k}^{(k+\beta-1,-\beta)}(2x^2-1),$$

$$R_n^{(k,\beta)}(x) = \frac{(n)!}{2^{k-1/2}\Gamma(n+k+\beta)} (x^2-1)^{k+\beta-1} P_n^{(k+\beta-1,-\beta)}(2x^2-1),$$

$$S_n^{(k,\beta)}(x) = \frac{\Gamma(n-k)}{2^{k-1/2}\Gamma(n-\beta)} x(1-x^2)^{k-\beta} P_{n-k-1}^{(k-\beta,\beta)}(2x^2-1).$$

These standardizations for $\beta = 1/2$ reduce to the standardized Gegenbauer polynomials used by Buschman when k is an even integer in [2]. Thus the results of [2] are particular cases of those given here when k is an even integer. A generalization, for the case when k is an odd integer in the standardizations used by Buschman, appears to be impossible. This is due to the fact that the solution to the integral equation in this case is expressible in terms of some other function as kernel.

Consider the integral equations

$$(1) \quad \int_x^1 F_n^{(k,\beta)}(t/x) g(t) dt = f(x),$$

$$(2) \quad \int_x^1 R_n^{(k,\beta)}(t/x) g_1(t) dt = f_1(x).$$

It is assumed that

(a) $f^m(1) = 0$, $f_1^m(1) = 0$, for $0 \leq m \leq 2k$,

(b) $f^{2k+1}(x)$, $f_1^{2k+1}(x)$ are piecewise continuous for $0 < a \leq x \leq 1$.

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The solutions of (1) and (2) are given by

$$(3) \quad g(t) = \int_t^1 G_n^{(k,\beta)}(t/y) y^{-2n+2k+2\beta-1} \left(-y^{-1} \frac{d}{dy}\right)^{2k+1} (y^{2n+2k-2\beta+1} f(y)) dy,$$

$$(4) \quad g_1(t) = \int_t^1 S_n^{(k,\beta)}(t/y) y^{-2n+2k-2\beta+2} \left(-y^{-1} \frac{d}{dy}\right)^{-1} (y^{2n+2k+2\beta-2} f_1(y)) dy,$$

respectively.

These integrals are in the form of convolutions with respect to the Mellin transformations. An application of this transform, use of the tables [3] and some manipulations yield the solutions given above.

The solution (3) of (1) can be verified directly by first proving that for $u > 1$

$$J(u) = \int_{1/u}^1 F_n^{(k,\beta)}(uv) G_n^{(k,\beta)}(v) dv = \{2^{2k}(2k)!\}^{-1} (u^2 - 1)^{2k} u^{2n-2k-2\beta+1}.$$

This integral can be written as

$$\int_{1/u}^1 (uv)^{2\beta} F_n^{(k,\beta)}(uv) \cdot v^{-2\beta} G_n^{(k,\beta)}(v) dv = u^{2\beta} J(u).$$

This can be rewritten in a standard form of a convolution for the Mellin transformation ([6] or [3, 6.1 (13)]):

$$\int_0^\infty (uv)^{2\beta} F_n^{(k,\beta)}(uv) [U(uv - 1)] v^{-2\beta} G_n^{(k,\beta)}(v) [1 - U(v - 1)] dv,$$

where $U(x) = 0, x < 0$; $U(x) = 1, x > 0$. Thus we have

$$\begin{aligned} & \mathfrak{M}\{u^{2\beta} J(u); s\} \\ &= \mathfrak{M}\{u^{2\beta} F_n^{(k,\beta)}(u) U(u - 1); s\} \mathfrak{M}\{u^{-2\beta} G_n^{(k,\beta)}(u) [1 - U(u - 1)]; 1 - s\}. \end{aligned}$$

From Rodrigue's formula [4, 10.8 (10)], we have

$$u^{2\beta} F_n^{(k,\beta)}(u) = \frac{u}{2^{n+k-1/2} \Gamma(k - \beta + n + 1)} \left(\frac{d}{u du}\right)^n \{u^{2n+2\beta} (u^2 - 1)^{n+k-\beta}\},$$

and

$$u^{-2\beta} G_n^{(k,\beta)}(u) = \frac{(-)^{n-k}}{2^{n-1/2} \Gamma(\beta + n)} \left(\frac{d}{u du}\right)^{n-k} \{(1 - u^2)^{n+\beta-1} u^{2n-2\beta-2k}\}.$$

Furthermore if the Mellin transformation of a certain function $\chi(x)$ is $\theta(s)$, then the Mellin transformation of $(d/xdx)^n \chi(x)$ will be

$(-)^n 2^n (s/2 - n)_n \theta(s - 2n)$. Using these results and the tables [3, 6.2 (31), (32), 6.1 (7)] we have, after some simplification,

$$\begin{aligned} \mathfrak{M}\{u^{2\beta} F_n^{(k,\beta)}(u) U(u-1); s\} \\ = \frac{1}{2^{k+1/2}} \frac{\Gamma(\frac{1}{2} + n - s/2) \Gamma(-\frac{1}{2} - s/2 - n - k)}{\Gamma(\frac{1}{2} - s/2) \Gamma(\frac{1}{2} - \beta - s/2)}, \\ \mathfrak{M}\{u^{-2\beta} G_n^{(k,\beta)}(u) [1 - U(u-1)]; 1-s\} \\ = \frac{1}{2^{k+1/2}} \frac{\Gamma(\frac{1}{2} - s/2) \Gamma(\frac{1}{2} - \beta - s/2)}{\Gamma(\frac{1}{2} + n - s/2) \Gamma(\frac{1}{2} - s/2 - n + k)}. \end{aligned}$$

Therefore,

$$\mathfrak{M}\{u^{2\beta} J(u); s\} = \frac{1}{2^{2k+1}(2k)!} B(-\frac{1}{2} - s/2 - n - k, 2k+1).$$

But from the tables [3, 6.2 (32), 6.1 (7)] it follows also that

$$[2^{2k}(2k)!]^{-1}(u^2 - 1)^{2k} u^{2n-2k+1} U(u-1)$$

has the same Mellin transform, hence the formula follows.

Substituting the value of $g(t)$ from (3) in (1), we have

$$\begin{aligned} I(x) = \int_x^1 F_n^{(k,\beta)}(t/x) \left(\int_t^1 G_n^{(k,\beta)}(t/y) y^{-2n+2k+2\beta-1} \left(-\frac{d}{ydy} \right)^{2k+1} \right. \\ \left. \cdot \{y^{2n+2k-2\beta+1} f(y)\} dy \right) dt. \end{aligned}$$

Now after changing the order of integration we obtain

$$\begin{aligned} I(x) = \int_x^1 y^{-2n+2k+2\beta-1} \left(-\frac{d}{ydy} \right)^{2k+1} \{y^{2n+2k-2\beta+1} f(y)\} \\ \cdot \left(\int_x^y F_n^{(k,\beta)}(t/x) G_n^{(k,\beta)}(t/y) dt \right) dy. \end{aligned}$$

If we write $v = t/y$, $u = y/x$, the inner integral becomes

$$yJ(u) = [2^{2k}(2k)!]^{-1}(y^2 - x^2)^{2k} y^{2n-2k-2\beta+2} \cdot x^{-2n+2\beta-2k-1}.$$

This gives

$$\begin{aligned} I(x) = - [2^{2k}(2k)!]^{-1} x^{-2n+2\beta-2k-1} \\ \int_x^1 (y^2 - x^2)^{2k} d \left[\left(-\frac{d}{ydy} \right)^{2k} \{y^{2n+2k-2\beta+1} f(y)\} \right]. \end{aligned}$$

Successive integrations by parts and application of the conditions $f^m(1) = 0$ gives $I(x) = f(x)$.

The second solution can be verified in a similar way.

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INFINITE ORDER DIFFERENTIAL EQUATIONS

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1. **Introduction.** Let $f(z) = \sum_{k=0}^{\infty} A_k z^k$ converge for $|z| < R$, where $0 < R \leq \infty$; let E be the vector space of entire functions of exponential type less than R ; and let $\mathfrak{D} = \sum_{k=0}^{\infty} A_k D^k$, where D is the differential operator. The purpose of this paper is to provide a brief derivation of the results of Muggli [2, p. 154] regarding the general solutions in E of the equations

$$(1) \quad \mathfrak{D}\phi = 0, \quad \text{and}$$

$$(2) \quad \mathfrak{D}\phi = \psi.$$

It will be shown that $\mathfrak{D}$ is a surjective endomorphism of E , reducing the problem of solving (2) to that of solving (1). It is easy to show that if ζ is a zero of f of order at least $h+1$ and of modulus less than R , then $z^h e^{\zeta z}$ is a solution of (1). If B is the set of all such exponential monomials, then Muggli's result says that B is a basis for the solutions of (1) and that each solution ϕ is representable as a sum of exponential monomials with exponent coefficients in the conjugate indicator diagram of ϕ . Each solution of (2) is then representable as the sum of a contour integral and a solution of (1).

Results similar to these have been obtained by Sheffer [3, p. 255]

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