A NOTE ON FREE LIE ALGEBRAS

G. LEGER[1]

In this paper we give two results on the following problem which was suggested to us by G. Hochschild: If L is a Lie algebra over a field K and if $H^2(L, M)=0$ for all L-modules M, is L a free Lie algebra? The universal mapping property of free Lie algebras plus the interpretation of $H^2(L, M)$ in terms of Lie algebra extensions shows that the converse is true. For details see Cartan-Eilenberg, *Homological algebra*, Chapters XIII and XIV. Here, Lie algebras need not have finite dimension.

THEOREM 1. *Let L be a Lie algebra over a field K such that*

1. *$H^2(L, K)=0$ where K is considered as a trivial L-module.*
2. *L has a vector subspace T such that $T\cap[L, L]=0$ and T generates L.*

Then L is a free Lie algebra.

PROOF. Suppose false and let $F=F(T)$ denote the free Lie algebra over the vector space T. The identity map: $T\rightarrow T$ can be extended (uniquely) to a homomorphism ϕ of F onto L with nonzero kernel R. R is contained in $[F, F]$. In fact, if x is in R, write $x=t+y$ with t in T, and y in $[F, F]$. Then $0=\phi(x)=t+\phi(y)$ so that t is in $T\cap[L, L]=0$.

Since F is a free Lie algebra, $\bigcap_i F^i=0$, where $F^1=F, \cdots, F^i=[F, F^{i-1}], \cdots$. Thus $[F, R]\neq R$ and we let R^* be an ideal of F of codimension one in R such that $R^*\supset[F, R]$. Then $L\approx F/R\approx(F/R^*)/(R/R^*)$ and, as a trivial L-module, $R/R^*\approx K$. Suppose F/R^* splits over R/R^*. Then F has a subalgebra S containing R^* such that $S/R^*+R/R^*=F/R^*$ and $S/R^*\cap R/R^*=0$. The latter property implies that S is a proper subalgebra of F. Now $S+R=F$ and $R\subset[F, F]=[S, S]+[S, R]+[R, R]\subset S+R^*\subset S$ so that $S=F$ and we have a contradiction.

THEOREM 2. *Suppose that L is generated by two elements and that L properly contains L^2 and L^2 properly contains L^3. Then, if $H^2(L, K)=0$, L is a free Lie algebra.*

PROOF. Let x and y generate L. Since $L\neq L^2$ we may assume $x\notin L^2$ without loss of generality. Let V be the vector space spanned

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by x and y; then if $V \cap L^2 = 0$ we are done by Theorem 1. If $z \in V \cap L^2$ and $z \neq 0$, then x and z still generate L. In this case $L = (x) + L^2$ so z has the form $z = \sum [a_i x + h_i, b_i x + h_i']$ with a_i, b_i in K and h_i, h_i' in L^2. This means that z is in L^3. But any element of L^2 is a sum of words in x and z and each word must contain z i.e., $L^2 \subseteq L^3$ contrary to hypothesis.

Harvard University and
Western Reserve University

A REMARK ON SEMI-SIMPLE LIE ALGEBRAS

GORDON BROWN

The following observation seems to be lacking in the literature.

Theorem. *The sum of the squares of the lengths of the roots of a semi-simple Lie algebra is equal to the dimension of a Cartan subalgebra.*

As usual the lengths of the roots are taken with respect to the non-degenerate Killing form Tr ad x ad y.

Proof. No generality is lost in assuming the ground field algebraically closed.

Consider the matrix M whose entries are the inner products (α, β), where α, β range over all the roots with respect to the Cartan subalgebra H. We have

$$(\alpha, \beta) = \operatorname{Tr} \operatorname{ad} h_\alpha \operatorname{ad} h_\beta = \sum\nolimits_\gamma \gamma(h_\alpha)\gamma(h_\beta) = \sum\nolimits_\gamma (\alpha, \gamma)(\gamma, \beta)$$

where h_α denotes the dual of α; i.e., $\alpha(h) = (h_\alpha, h)$ for all h in H. Thus $M^2 = M$ and Tr M = rank M. Hence $\sum_\alpha (\alpha, \alpha) = \dim H$.

University of Illinois

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