LINEAR MODULUS OF A LINEAR OPERATOR

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1. Introduction. Let T be a continuous linear operator on real or complex $L_1(S, \mathfrak{F}, \mu)$, where $(S, \mathfrak{F}, \mu)$ is a σ -finite measure space.² Theorem 1 states that a positive operator $|T|$ without increased norm "majorizes" T , i.e., that $|Tf(s)| \leq |T| |f(s)|$ holds. As an application, a theorem of Dunford and Schwartz [5] and an ergodic theorem are proved. If L_1 and T are real the existence of $|T|$ is obtained from lattice properties of the Banach space of all T on L_1 . Among the properties of the mapping $T \rightarrow |T|$ an important one is that it preserves invariant sets. An application which generalizes Jacobs' ergodic decomposition of positive T to general T will be given elsewhere by the second author.³

2. Existence. The main result is:

THEOREM 1.⁴ *If T is a continuous linear operator on L_1 then there exists a positive linear operator $|T|$, called the linear modulus of T , such that*

$$(a) \quad |Tf(s)| \leq |T| |f(s)|,$$

$$(b) \quad |||T|||_1 \leq \|T\|_1.$$

For $f \in L_1^+ = \{f \in L_1 \mid f \geq 0\}$, $|T|$ is given by

$$(2.1) \quad |T|f(s) = \sup_{|g| \leq f} |Tg|(s).$$

PROOF. θ shall denote the set of all finite disjoint and measurable decompositions $D = \{S_1(D), \dots, S_{k(D)}(D)\}$ of S . θ is partially ordered with respect to refinement and is a decreasingly filtered system. Let N be the class of simple functions, i.e., $h \in N$ if $h = \sum_{i=1}^n a_i(h) \psi_{E_i(h)}(s)$, where ψ_{E_i} is the characteristic function of $E_i \in \mathfrak{F}$. We define for $f \in L_1^+$:

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² All equations are understood to hold almost everywhere.

³ See: *Über den Absolutbetrag stetiger linearer Operatoren und seine Anwendung auf ergodische Zerlegungen*, Math. Scand. **13** (1964).

⁴ Note that $|T|$, $|f|$, $\|f\|$, $\|T\|$, $|||T|||$ have different meanings.

$$(2.2) \quad Q(D, T, f)(s) = \sum_{i=1}^{k(D)} |T(f\psi_{S_i(D)}(s))|.$$

As $Q(D, T, f)(s)$ is monotone increasing on θ for fixed f and

$$(2.3) \quad \|f\|_1 = \sum_{i=1}^{k(D)} \|f\psi_{S_i(D)}\|_1$$

implies norm boundedness, that is

$$(2.4) \quad \|Q(D, T, f)\|_1 \leq \|T\|_1 \|f\|_1,$$

we may define

$$(2.5) \quad |T|f(s) = \lim_{D \in \theta} Q(D, T, f)(s) \in L_1^+.$$

If $f_1, f_2 \in N \cap L_1^+$ we may assume $f_i(s) = \sum_{k=1}^n a_{ik} \psi_{E_k}(s)$ with equal E_k for f_1 and f_2 and thus, for D fine enough so that the E_k occur among the $S_i(D)$ or finer, we get $Q(D, T, f_1) + Q(D, T, f_2) = Q(D, T, f_1 + f_2)$. This proves linearity on $N \cap L_1^+$ and, by approximation, on L_1^+ . Now $|T|$ can be extended by linearity on L_1 and from (2.4) we get $\||T|\|_1 \leq \|T\|_1$. To prove (2.1) we remark that $|T|f \geq |Tg|$ follows from the definitions if $g \in N$ and $|g| \leq f$. Letting Lf denote the right side of (2.1), it is clear that $|T|f \geq Lf$. If the converse inequality does not hold, then there exists an $f \in N \cap L_1^+$, an $\epsilon > 0$, a $D \in \theta$ and an $A_0 \in \mathfrak{F}$ with $\mu(A_0) > 0$ and

$$(2.6) \quad Q(D, T, f)(s) \geq Lf(s) + \epsilon \quad \text{on } A_0.$$

Now for $i=1$ we can find a complex number a_i with $|a_i| = 1$, such that

$$(2.7) \quad |T(a_i f \psi_{S_i(D)}) - |T(f \psi_{S_i(D)})|| < \epsilon/2k(D)$$

holds on a subset $A_i \subseteq A_{i-1}$ with $\mu(A_i) > 0$. We do the same for $i=2, \dots, k(D)$. Setting $g = \sum_{i=1}^{k(D)} a_i f \psi_{S_i(D)}$ we have $|g| \leq |f|$ and this leads to a contradiction to (2.6) on $A_{k(D)}$. From (2.1) follows (a) and $\||T|\|_1 \geq \|T\|_1$.

REMARKS.

(1) These results carry over if T is an operator from $L_1(S, \mathfrak{F}, \mu)$ into $L_1'(S', \mathfrak{F}', \mu')$.

(2) It is not difficult to formulate an analogue of Theorem 1 for the space of finite countably additive set functions on $\mathfrak{F}$ which are absolutely continuous with respect to μ .

3. Lattice properties for real T . In this section we consider real L_1 and real operators T .

THEOREM 2. *The linear space $\mathfrak{F}$ of all real continuous linear operators T on L_1 is a lattice.*

PROOF. f^+ and f^- shall denote $f \vee 0$ and $(-f) \vee 0$, so that $f^+ - f^- = f$ and $f^+ + f^- = |f|$. We define for $f \in L_1^+$:

$$(3.1) \quad \begin{aligned} M(D, T, f)(s) &= \sum_{i=1}^{k(D)} (T(f \cdot \psi_{s_i(D)}))^+(s), \\ N(D, T, f)(s) &= \sum_{i=1}^{k(D)} (T(f \cdot \psi_{s_i(D)}))^{-}(s). \end{aligned}$$

We get $0 \leq M(D, T, f)$, $N(D, T, f) \leq Q(D, T, f)$ and, since the functions are monotone on θ , we may define

$$(3.2) \quad T^+ f = \lim_{D \in \theta} M(D, T, f), \quad T^- f = \lim_{D \in \theta} N(D, T, f).$$

Linearity on L_1^+ is proved as for $|T|$ and by extension we get linear operators T^+ and T^- on L_1 with $\|T^+\|_1 \leq \|T\|_1$ and $\|T^-\|_1 \leq \|T\|_1$. To prove for example $T^+ = T \vee 0$ we first state that $T^+ \geq 0$, $T^+ \geq T$. If $R \geq T$ and $R \geq 0$ then we have $R^- = 0$ and from $M(D, R, f) \geq M(D, T, f)$ it follows that $R \geq T^+$. In fact now we have

$$T^+ + T^- = |T|, \quad T^+ - T^- = T.$$

4. **Applications to ergodic theorems.** The ordered pair $(p_1(s), p_2(s))$ of measurable functions is said to be *admissible with respect to the operator T* if $|f(s)| \leq p_1(s)$ and $f \in L_1$ imply that $|Tf(s)| \leq p_2(s)$. We note that (p_1, p_2) will be admissible with respect to T if p_1 is arbitrary and p_2 is defined by

$$p_2(s) = \sup_{|f| \leq p_1} |Tf(s)|.$$

The following theorem is a consequence of (2.1):

THEOREM 3. *If T is a continuous linear operator on L_1 then the admissible pairs of T are admissible with respect to $|T|$.*

COROLLARY 1 [5]. *If T is a continuous linear operator on L_1 whose L_∞ -norm is finite then there exists a positive linear operator P on L_1 whose L_1 and L_∞ -norms do not exceed those of T and which is such that*

$$(4.1) \quad |T^n f(s)| \leq P^n f(s).$$

PROOF. With $P = |T|$, (4.1) follows by induction from (2.a) and $\|P\|_\infty \leq \|T\|_\infty$ from (2.1).

COROLLARY 2. If T is a linear operator on L_1 with $\|T\|_1 \leq 1$ and $f, p \in L_1$, $p \geq 0$ then

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n T^k f(s)}{\sum_{k=0}^n |T|^k p(s)}$$

exists and is finite almost everywhere on $\{s: \sum_{k=0}^{\infty} |T|^k p(s) > 0\}$.

PROOF. Theorem 3 of [3] is applied to the sequence $p_k(s) = |T|^k p(s)$.

THEOREM 4 [3]. Let T be a linear operator on L_1 with $\|T\|_1 \leq 1$ and let p_n be a sequence of measurable non-negative functions such that $|f(s)| \leq p_j(s)$ implies $|Tf(s)| \leq p_{j+1}(s)$. Then:

$$\lim_{n \rightarrow \infty} \frac{T^n g(s)}{p_0(s) + \cdots + p_n(s)} = 0$$

on $\{s: \sum_{k=0}^{\infty} p_k(s) > 0\}$, for each $g \in L_1$.

PROOF. We may consider $A_k = \{s: p_k(s) > 0\}$ only. Take $p \in L_1$ with $p > 0$ on A_k and $0 \leq p \leq p_k$ and take $f = |T|^k |g|$. Theorem 1 of [1] implies that

$$\lim_{m \rightarrow \infty} \frac{|T|^m f(s)}{\sum_{i=0}^m |T|^i p(s)} = 0$$

on A_k . If $m+k=n$ the theorem follows from $|T^n g| \leq |T|^n |g| = |T|^m f$ and $p_0 + \cdots + p_n \geq \sum_{i=0}^m |T|^i p$.

5. **Simple properties of $T \rightarrow |T|$.** We give a list of properties of the mapping $T \rightarrow |T|$:

THEOREM 5.

$$(5.1) \quad |T|^n |f| \geq |T^n f| \quad \text{for all } f \in L_1,$$

$$(5.2) \quad |T| \geq 0, T \geq 0 \text{ implies } T = |T|,$$

$$(5.3) \quad |aT| = |a| |T|,$$

$$(5.4) \quad |T_1 + T_2| \leq |T_1| + |T_2|,$$

$$(5.5) \quad |||T|||_1 = \|T\|_1,$$

$$(5.6) \quad \|T\|_{\infty} = K < \infty \text{ implies } |||T|||_{\infty} = K,$$

$$(5.7) \quad T \rightarrow |T| \text{ is norm continuous,}$$

(5.8) $g(s) \in L_\infty$ and $T'f = gTf$ for all $f \in L_1$ implies $|T'| = |g| |T|$.

(5.9) If $S_i \subseteq S$ are disjoint sets ($i = 1, \dots, n$) such that $T_i f = T(\psi_{S_i} f)$

or $T_i f = \psi_{S_i} T_i^* f$ for some T_i^* then $\sum_{i=1}^n |T_i| = \left| \sum_{i=1}^n T_i \right|$,

(5.10) $|T_1 T_2| \leq |T_1| |T_2|$ and " $<$ " occurs.

The proofs follow at once. Concerning (5.10) we remark that if S consists of n points, T can be described by a matrix (a_{ik}) and from (5.9) we get that $|T|$ is described by $(|a_{ik}|)$. Now

$$T_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

give an example with $|T_1 T_2| < |T_1| |T_2|$.

REMARK. (5.5) and (5.6) would lead one to expect that other norms are preserved as well. This is not true in general, however, as the following example shows: Let S consist of two points, each of them of measure one. Functions are pairs (a, b) . With $T(a, b) = (a - b, a + b)$ we get $\|T\|_1 = 2$, $\|T\|_\infty = 2$ and $|T|(a, b) = (a + b, a + b)$. The application of $|T|$ to $(1, 1)$ shows that $\||T|\|_2 \geq 2$ and it equals 2 by norm convexity. But for $(a, b) \neq (0, 0)$:

$$\frac{\|T(a, b)\|_2}{\|(a, b)\|_2} = \frac{((a - b)^2 + (a + b)^2)^{1/2}}{(a^2 + b^2)^{1/2}} = \sqrt{2} \quad \text{yields } \|T\|_2 = \sqrt{2}.$$

6. Simple operators. We assume L_1 to be separable in this section.

THEOREM 6. *There exists a linear space $\mathfrak{F}_s$ of operators on L_1 which is dense in the strong operator topology in the space $\mathfrak{F}$ of all linear continuous operators on L_1 , such that for $T \in \mathfrak{F}_s$, $|T|$ is uniquely determined by (5.1)–(5.4) and can explicitly be calculated.*

PROOF. We define $\mathfrak{F}_s$ as the set of "simple operators" T of the form:

$$(6.1) \quad Tf(s) = \sum_{i=1}^n \sum_{k=1}^{r_i} a_{ik} \int_{S_i} f d\mu \cdot \psi_{A_{ik}},$$

where the a_{ik} are complex, the S_i are disjoint and the A_{ik} are disjoint for every fixed i and have finite measure. It follows that $\|T\|_1 = \max_i \left\{ \sum_{k=1}^{r_i} |a_{ik}| \cdot \mu(A_{ik}) \right\} < \infty$. $|T|$ has by (5.9) the same form with $|a_{ik}|$ instead of a_{ik} . To prove that $T + T' \in \mathfrak{F}_s$ for $T, T' \in \mathfrak{F}_s$, assume $S_i = S'_i$ by considering refinements. Let $T \rightarrow T'$ satisfy (5.1)–(5.4) and $T \in \mathfrak{F}_s$. T has the form $\sum_{i,k} T_{ik}$ with $T_{ik} = |T_{ik}| \cdot a_{ik} / |a_{ik}|$. From $|T|$ we know that it is minimal with (5.1) because of

(2.1). It then follows that $|T| \leq T \leq \sum_{i,k} T_{ik} (= \sum_{i,k} |T_{ik}| = |T|)$ and hence equality holds. The strong operator density of $\mathfrak{F}_s$ in $\mathfrak{F}$ is proved by the usual ϵ, δ method using the following idea: Given $T \in \mathfrak{F}$ let $\{f^{(1)}, f^{(2)}, \dots\}$ be dense in L_1 , the $f^{(m)}$ simple functions. To construct $T^{(r)} \in \mathfrak{F}_s$ consider $\{f^{(1)}, f^{(2)}, \dots, f^{(r)}\}$ only. They have the form $f^{(m)} = \sum_k c_{mk} \psi_{S_k}$ with disjoint and common S_k and $\mu(S_k) < \infty$. Approximate $T\psi_{S_k}$ by a simple function $g(T\psi_{S_k}) = \sum_{i=1}^k d_{ik} \psi_{A_{ik}}$ such that $\|g(T\psi_{S_k})\|_1 \leq \|T\psi_{S_k}\|_1$, the A_{ik} disjoint for fixed k and all $|c_{mk}| \|T\psi_{S_k} - g(T\psi_{S_k})\|_1 < \delta(r)$. Let $T^{(r)}$ be the simple operator which maps ψ_{S_k} on $g(T\psi_{S_k})$, so that $\|T^{(r)}\|_1 \leq \|T\|_1$. We choose $\delta(r)$ small enough so that $\|T^{(r)}f^{(m)} - Tf^{(m)}\|_1 < \epsilon(r)$ for $1 \leq m \leq r$. If $\epsilon(r)$ is a null sequence the $T^{(r)}$ converge to T in the strong operator topology.

7. A counter-example for strong operator continuity. From §6 the question arises whether the mapping $T \rightarrow |T|$ is continuous in the strong operator topology. The following example shows that this is not true even in weak operator topology. We define $S_i = \{1, 2\}$, $\mathfrak{F}_i =$ all subsets of S_i , $\mu_i\{1\} = \mu_i\{2\} = \frac{1}{2}$, $S = \prod_{i=1}^{\infty} S_i$. Let $\mathfrak{F}_{(k)}$ be generated by the sets $A_{i_1 \dots i_k} = \{\{i_1\} \times \{i_2\} \times \dots \times \{i_k\} \times S_{k+1} \dots\}$ and $\mathfrak{F}$ by the $\mathfrak{F}_{(k)}$, μ : product measure. Define T_n on L_1 by

$$(7.1) \quad \begin{aligned} T_n f = & \sum_{i_1 + \dots + i_n = 1 \bmod 2} \int_S f \cdot \psi_{A_{i_1 \dots i_n}} d\mu \\ & - \sum_{k_1 + \dots + k_n = 0 \bmod 2} \int_S f \cdot \psi_{A_{k_1 \dots k_n}} d\mu. \end{aligned}$$

It follows that $|T_n|f = \int_S f d\mu$ and thus $|T_n| = \text{const} \neq 0$. From (7.1) it follows that for $\mathfrak{F}_{(k)}$ measurable $f \in L_1$ and $l > k$ we have $T_l f = 0$. Such f are dense in L_1 , and $\|T_n\|_1 = 1$ and thus we see that T_n converges in the strong operator topology to 0.

8. Invariance. The concept of *positive* T is not invariant with respect to isometries R of L_1 , i.e., for positive T , $R^{-1}TR$ need not be positive. But this invariance holds and carries over to the mapping $T \rightarrow |T|$ if R comes from a point isomorphism: Let R denote a point isomorphism of $(S, \mathfrak{F}, \mu)$ and $(S', \mathfrak{F}', \mu')$, i.e., the mapping $s \rightarrow R(s) = s'$ shall be 1-1, measurable as well as R^{-1} , onto and $\mu'(E') = \mu(R^{-1}E')$. R induces the isometric mapping $R'f'(s) = f'(Rs)$ of L'_1 onto L_1 .

PROPOSITION 1. $|R'^{-1}TR'| = R'^{-1}|T|R'$.

PROOF. As R' and R'^{-1} come from a point transformation we may write $|R'^{-1}h(s)| = R'^{-1}|h(s)|$. Further, $R^{-1}(\theta') = \theta$ and we get for $f' \in L'_1$ and $g(s) = R'f'(s)$ the equations

$$\begin{aligned}
|R'^{-1}TR'|f'(s') &= \lim_{D' \in \theta'} \sum_{i=1}^{k(D')} |R'^{-1}TR'(f' \cdot \psi_{S_i(D')}(s'))| \\
&= R'^{-1} \lim_{D' \in \theta'} \sum_{i=1}^{k(D')} |T(g(s) \psi_{R^{-1}(S_i(D'))}(s))| \\
&= R'^{-1} |T| g(s) = R'^{-1} |T| R'f'(s').
\end{aligned}$$

The proof generalizes to every R' induced by an algebra isomorphism.

9. Invariant sets. For positive T with $\|T\|_1 = 1$, S splits into two (up to null sets) uniquely determined sets $D(T)$ and $C(T)$, the dissipative and the conservative parts of S (see e.g. [4, Theorem 2]). For arbitrary T with $\|T\|_1 = 1$ we define $D(T) = D(|T|)$ and $C(|T|) = C(T)$. A set $I \in \mathfrak{F}$ is called weakly T -invariant if $\{s: Tf(s) \neq 0\} \subseteq I$ for all $f \in L_1$ with $\{s: f(s) \neq 0\} \subseteq I$, it is called T -invariant if furthermore $I \subseteq C(T)$. An easy consequence of (2.1) now is:

PROPOSITION 2. *The (weakly) T -invariant sets are exactly the (weakly) $|T|$ -invariant sets.*

As the $|T|$ -invariant sets form a σ -field according to Lemma 5 of [4] we get

COROLLARY 3. *The T -invariant sets form a σ -field.*

These considerations are extended to the case in which $T \neq 0$, by defining $C(T) = C(|T|/\|T\|_1)$ for them. This is not quite what one would wish to mean by conservative but it seems more useful than considering all T with $\|T\|_1 \leq 1$, as $\|T\|_1 < 1$ would always give empty conservative parts.

REMARK. The main result was obtained by both authors; in §§3 and 5-9 the second author gives some applications and considers related problems.

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