n-FRAMES IN EUCLIDEAN k-SPACE

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1. **Introduction.** An n-frame F_n is a union of n arcs, $F_n = \bigcup_{i=1}^{n} A_i$, with a distinguished point p such that, if $n=1$, p is an end point of A_1, and if $n>1$, p is an end point of each A_i and $A_i \cap A_j = p$, $i \neq j$. We introduce the distinguished point in order to differentiate between a 1-frame and a 2-frame. A 1-frame is an arc with an end point distinguished and a 2-frame is an arc with an interior point distinguished. This difference will keep certain logical difficulties from arising in the inductive proof of Theorem 1_n.

In E^k let B_i be the arc in the x_1, x_2 plane, defined in polar coordinates by $r \leqq 1$, $\theta = \pi(1-1/i)$. For n a positive integer, the *standard n-frame* G_n is defined by $G_n = \bigcup_{i=1}^{n} B_i$. An n-frame F_n in E^k is said to be *tame* if there is a homeomorphism of E^k onto itself which carries F_n onto G_n. Otherwise F_n is said to be *wild*. For $n>1$, F_n is said to be *mildly wild* if it is wild and $F_n - (A_i - p)$ is tame for $i = 1, 2, \cdots, n$. In [3] it was shown that for each $n>1$ there are mildly wild n-frames in E^3. Since there are wild arcs in E^k for each $k>3$ [1], there will be wild n-frames for these dimensions. However, we will show that there are no mildly wild n-frames in E^k for $k>3$. It then follows that, for $k>3$, the union of two tame arcs meeting only in a common end point is a tame arc. With a small amount of additional argument we will show that a wild arc (simple closed curve) in E^k, $k>3$, must fail to be locally flat at each point of some Cantor set. (If S is an arc (simple closed curve) in E^k, we say that S is locally flat at $p \in S$ if there is a neighborhood U of p and a homeomorphism h which carries U onto E^k with $h(U \cap S)$ lying in the x_1-axis.)

Through the remainder of this paper we will assume that we are working in an euclidean space E^k with $k>3$. We recall that for an arc or simple closed curve in E^k to be tame it is sufficient that it be locally flat at each of its points. This result for simple closed curves is proved in [5]. The same technique of proof may be used to establish the corresponding result for arcs.

2. **Basic lemmas.** In [4] it was stated that the result contained in Lemma 2 below followed as a corollary to a theorem concerning manifolds with boundary E^{k-1} and interior E^k. Since it seems that a more

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direct proof should be available to the reader, an alternate proof is included in this paper.

LEMMA 1. *Let L be an arc in E^k, p an end point of L, and U a neighborhood of $L-p$. If L is locally flat at each point of $L-p$, then there is a homeomorphism f of E^k onto itself such that f is the identity outside U and $f(L)$ is locally polyhedral at each point of $f(L-p)$.*

PROOF. Let p_0 be the end point of L, different from p, and let $p_1, p_2, \cdots$ be a sequence of points of L converging to p with $p_0 < p_1 < p_2 < \cdots$ relative to the order of L from p_0 to p. For each integer i let $\epsilon_i = 1/i$ and let A_i be the closed subarc of L from p_0 to p_i. Since A_2 is tame, we may select a closed k-cell neighborhood U_1 of A_1 with the properties: (1) U_1 is contained in the ϵ_1-neighborhood of A_1 and in U, (2) $U_1 \cap (L-A_2) = \square$, and (3) U_1 may be assigned a combinatorial triangulation in which A_1 is polyhedral. We then apply Homma's Theorem [5] to obtain a homeomorphism f_1 of E^k onto itself such that f_1 is the identity outside U_1 and $f_1(A_1)$ is polyhedral in E^k.

Assume that for each integer $i > 1$ certain homeomorphisms $f_{i-1}, \cdots, f_1$ of E^k onto itself have been constructed so that $f_{i-1} \cdots f_2 f_1(A_{i-1})$ is polyhedral in E^k. If $i=2$, let U_2 be a closed k-cell neighborhood of $\mathrm{Cl}\, f_1(A_2 - A_1)$ with the properties: (1) U_2 is contained in the ϵ_2-neighborhood of $f_1(A_2 - A_1)$ and in U, (2) $U_2 \cap f_1(L - A_3) = \square$, and (3) U_2 may be assigned a combinatorial triangulation in which $U_2 \cap f_1(A_2)$ appears as a polyhedron. We then apply Theorem 2.1 of [5] to obtain a homeomorphism f_2 of E^k onto itself such that f_2 is the identity outside U_2 and on $f_1(A_1)$, and $f_2 f_1[\mathrm{Cl}(A_2 - A_1)]$ is polyhedral in E^k. Note that at this point $f_2 f_1(A_2)$ is polyhedral in E^k. If $i > 2$, since A_{i+1} is tame, we may select a closed k-cell neighborhood U_i of $\mathrm{Cl}\, f_{i-1} \cdots f_2 f_1(A_i - A_{i-1})$ with the properties:

(1) U_i is contained in the ϵ_i neighborhood of $f_{i-1} \cdots f_2 f_1(A_i - A_{i-1})$ and in U,

(2) $U_i \cap [f_{i-1} \cdots f_2 f_1(L - A_{i+1})] = \square$,

(3) $U_i \cap [\bigcup_{j=1}^{i-2} U_j] = \square$,

(4) $U_i \cap [\bigcup_{j=1}^{i-2} f_{i-1} \cdots f_2 f_1(U_j)] = \square$, and

(5) U_i may be assigned a combinatorial triangulation in which $U_i \cap f_{i-1} \cdots f_2 f_1(A_i)$ is polyhedral. Again Theorem 2.1 of [5] is applied to obtain a homeomorphism f_i of E^k onto itself such that f_i is the identity outside U_i and on $f_{i-1} \cdots f_2 f_1(A_{i-1})$, and

$$f_i[\mathrm{Cl}\, f_{i-1} \cdots f_2 f_1(A_i - A_{i-1})]$$

is polyhedral in E^k.

For each $x \in E^k$ we set $f(x) = \lim_{i\to\infty} f_i \cdots f_2 f_1(x)$. Depending principally on the fact that if $x \notin \bigcup_{j=1}^{\infty} U_j$, $f(x) = x$, and if $x \in U_j$, $f(x) = f_{j+1} f_j f_{j-1}(x)$ one establishes that f is a homeomorphism of E^k onto itself, and $f(L)$ is locally polyhedral at each point different from $f(p)$.

LEMMA 2. *If L is as in Lemma 1, then L is tame.*

PROOF. Let f be a homeomorphism of E^k onto itself so that $f(L)$ is locally polyhedral at each point of $f(L) - f(p)$. We use Lemma 2 of [2] to obtain a homeomorphism g of E^k onto itself such that $gf(L)$ is polyhedral. We then use Theorem 5 of [6] to obtain a homeomorphism h of E^k onto itself that carries $gf(L)$ onto B_1.

For each positive integer n we may establish the following theorem.

THEOREM 1_n. *Let $F_n = \bigcup_{i=1}^{n} A_i$ be an n-frame, with distinguished point p, such that A_i, $i = 1, 2, \cdots, n$, is locally flat at each point of $A_i - p$. Then F_n is tame.*

PROOF. Theorem 1_1 has been proved in Lemma 2. We next assume that Theorem 1_{n-1}, $n \geqq 2$, is true and proceed to show that Theorem 1_n is true.

Let $F_n = \bigcup_{i=1}^{n} A_i$ be an n-frame which satisfies the hypotheses of Theorem 1_n. There is a homeomorphism ϕ_1 of E^k onto itself such that $F_{n-1} = F_n - (A_n - p)$ is carried onto G_{n-1}. Since $\phi_1(A_n)$ is tame, there is a neighborhood V of $\phi_1(A_n - p)$ which does not intersect G_{n-1}, and, by Lemma 1, a homeomorphism ϕ_2 on E^k such that $\phi_2\phi_1(A_n)$ is locally polyhedral at each point of $\phi_2\phi_1(A_n - p)$ and ϕ_2 is fixed outside V. We next construct a homeomorphism ϕ_3 on E^k such that $\phi_3\phi_2\phi_1(A_n) = B_n$ and ϕ_3 is fixed on G_{n-1}. The homeomorphism $\phi_3\phi_2\phi_1$ will then carry F_n onto G_n and the proof of Theorem 1_n will be complete. Since the construction of ϕ_3 is almost identical with that used in the proof of Lemma 2 of [2], we will only give an outline of the construction.

We use the local connectivity of $\phi_2\phi_1(A_n)$ to find a sequence $\{V_m\}_{m=1}^{\infty}$ of closed cubical neighborhoods of the origin such that (1) the end points of B_n and $\phi_2\phi_1(A_n)$, different from the origin, are outside V_1, (2) the diameters of the V_m converge to zero, and (3) if L is any subarc of $\phi_2\phi_1(A_n)$ whose end points lie in V_m, then L is contained in Int V_{m-1}. We will further assume that $\phi_2\phi_1(A_n) \cap \mathrm{Bd}\, V_{2m}$ is a finite set of points and that no pair of components of $\phi_2\phi_1(A_n) - V_{2m}$ share a common end point.

For each positive integer m let $u_{m1}, \cdots, u_{mk(m)}$ be the components of $\phi_2\phi_1(A_n) - \mathrm{Int}\, V_{2m}$ which have both end points on Bd V_{2m}. Since there can be no knotting or linking of polyhedral simple closed curves in an euclidean space of dimension greater than three, each of the u_{mi} may be moved into Int V_{2m} without moving points outside $V_{2m-1}-V_{2m+1}$ or on G_{n-1}. Thus we may construct a semilinear homeomorphism f_m such that $f_m\phi_2\phi_1(A_n)\cap \text{Bd } V_{2m}$ is a single point, and f_m is the identity on $E^k-(V_{2m-1}-V_{2m+1})$ and on G_{n-1}. A homeomorphism f is defined by $f(x)=x$, if $x\in E^k-V_1$, $f(x)=f_m(x)$, if $x\in V_{2m-1}-V_{2m+1}$, $m=1, 2, \cdots$, and f carries the origin onto itself.

Let 0 denote the origin, let x_0 be the end point of B_n different from 0, $x_m=B_n\cap \text{Bd } V_{2m}$, $m=1, 2, \cdots$, y_0 the end point of $f\phi_2\phi_1(A_n)$ different from 0, and $y_m=f\phi_2\phi_1(A_n)\cap \text{Bd } V_{2m}$, $m=1, 2, \cdots$. Let g_0 be a semilinear homeomorphism on E^k such that g_0 is fixed on V_1 and $g_0(y_0)=x_0$. For $m=1, 2, \cdots$, let g_m be a semilinear homeomorphism on E^k such that g_m is fixed outside $V_{2m-1}-V_{2m+1}$ and on G_{n-1}, and $g_m(y_m)=x_m$. A homeomorphism g is then defined by $g(x)=g_0(x)$, for $x\in E^k-V_1$, $g(x)=g_m(x)$, for $x\in V_{2m-1}-V_{2m+1}$, and $g(0)=0$.

Again since there can be no knotting or linking of polyhedral simple closed curves in E^k, we may construct homeomorphisms h_m with the following properties. The map h_0 is fixed on V_2 and on G_{n-1}, and carries the subarc of $gf\phi_2\phi_1(A_n)$ from x_0 to x_1 onto the linear segment $[x_0x_1]^-$ For $m=1, 2, \cdots$, h_m is fixed on $E^k-(V_{2m}-V_{2m+2})$ and on G_{n-1}, and carries the subarc of $gf\phi_2\phi_1(A_n)$ from x_m to x_{m+1} onto the linear segment $[x_mx_{m+1}]^-$ We set $h(x)=h_0(x)$, if $x\in E^k-V_2$, $h(x)=h_m(x)$, if $x\in V_{2m}-V_{2m+2}$, and $h(0)=0$. Finally we take $\phi_3=hgf$.

COROLLARY 1_n. *There are no mildly wild n-frames in E^k.*

PROOF. Suppose $F_n=\bigcup_{i=1}^n A_i$, $n>1$, is an n-frame such that for each $j=1, 2, \cdots, n$, $F_n-(A_j-p)$ is tame. Then each A_j is tame and, by Theorem 1_n, F_n is tame.

COROLLARY 2_2. *If A_1 and A_2 are tame arcs in E^k, meeting only in a common end point, then $A_1\cup A_2$ is tame.*

THEOREM 2. *If A is a wild simple closed curve (arc) in E^k and E is the set of points at which A fails to be locally flat, then E contains a Cantor set.*

PROOF. For A a simple closed curve, we know that E is nonempty [5]. By the definition of local flatness, the set of points at which A is locally flat is an open subset of A, and E is therefore closed. If we establish that E has no isolated points, there are two possibilities. First, E may be totally disconnected, in which case E is a Cantor set. Secondly, E may have a nondegenerate component K, in which case K is either an arc or $K=A$.

In order to show that there are no isolated points of E, let us consider a point q of A such that there is a neighborhood U of q, relative to A, with A locally flat at each point of $U-q$. We select two arcs A_1 and A_2 of A such that $A_1 \cup A_2 \subset U$ and $A_1 \cap A_2 = q$. By Lemma 2 and Corollary 2_2, $A_1 \cup A_2$ is tame, and A is therefore locally flat at q.

This proves Theorem 2 for A a simple closed curve. A similar argument establishes the theorem for A an arc.

3. **Added in proof.** Suppose that M is a finite 1-complex topologically embedded in E^k, $k>3$, V is the set of vertices of M, and M is locally flat at each point of $M-V$ (equivalently, each 1-simplex of K is locally flat at each of its interior points). By first applying Homma's Theorem, as in Lemma 1, we may construct a homeomorphism f of E^k onto itself such that $f(M)$ is locally polyhedral at each point of $f(M)-f(V)$. Then, by applying the technique of proof used in Theorem 1_n at each point of $f(V)$, we may construct a homeomorphism g of E^k onto itself such that $gf(M)$ is polyhedral. Thus we see that M is tame. An immediate corollary is that a finite 1-complex, topologically embedded in E^k, $k>3$, is tame if and only if each simplex is tame.

References

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