UNIONS OF RELATIONAL SYSTEMS

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We shall assume throughout that K is a class of structures (i.e., relational systems) that is elementary in the wider sense; that is, K is the class of all models of some finite or infinite set of sentences of the first order predicate logic with identity. A structure $\mathfrak{B}$ is said to be a *union of* a set M of structures if each $\mathfrak{A}\in M$ is a substructure of $\mathfrak{B}$ and every element of $\mathfrak{B}$ is an element of some $\mathfrak{A}\in M$. Our purpose in this paper is to prove the result below.

The following two conditions are equivalent:

(a_1) *if $\mathfrak{B}$ is a union of some subset M of K, then $\mathfrak{B}\in K$*;

(b_1) *K is the class of all models of some set of sentences of the form*

$$\forall v_0 \exists v_1 \exists v_2 \cdots \exists v_m \Phi,$$

where Φ has no quantifiers.

We shall actually consider the more general notion of an n-union, where n is a natural number. $\mathfrak{B}$ is said to be an *n-union of* a set M of structures if each $\mathfrak{A}\in M$ is a substructure of $\mathfrak{B}$ and each collection of at most n elements of $\mathfrak{B}$ is included in some $\mathfrak{A}\in M$. Thus $\mathfrak{B}$ is a 1-union of M iff $\mathfrak{B}$ is a union of M; moreover, $\mathfrak{B}$ is a 0-union of M iff M is nonempty and $\mathfrak{B}$ is a common extension of all $\mathfrak{A}\in M$. Note that if $\mathfrak{B}$ is an n-union of M, then $\mathfrak{B}$ is an m-union of M for all $m\leq n$. Our main result will be

THEOREM A. *If n is a fixed natural number, then the following two conditions are equivalent*:

(a_n) *if $\mathfrak{B}$ is an n-union of some subset M of K, then $\mathfrak{B}\in K$*;

(b_n) *K is the class of all models of some set of sentences of the form*

$$\forall v_0 \cdots \forall v_{n-1} \exists v_n \cdots \exists v_m \Phi,$$

where Φ has no quantifiers and $n-1\leq m$.

Theorem A above was announced without proof in 1959 in [**5**]. The original proof was by methods similar to those introduced in [**4**]. Since that time, continuing developments in the theory of models has made it possible to give progressively shorter proofs of the theorem, one of which is given here. The present proof depends on the notion of a special structure, which is due to Morley and Vaught [**8**]. The author is indebted to C. C. Chang and Roger Lyndon for helpful

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discussions in connection with the results of this paper.

Before passing to the proof of Theorem A, we shall pause to mention some closely related earlier results. The following theorem is due to Tarski [12] and Łoś [6].

The following two conditions are equivalent:

(a′) *if* $\mathfrak{B}$ *is a substructure of some* $\mathfrak{A}\in K$, *then* $\mathfrak{B}\in K$;

(b′) *K is the class of all models of some set of universal sentences.*

A refinement of the above theorem is the result below, which is due to Vaught and is implicit in [13].

If n is a fixed natural number, then the following two conditions are equivalent:

(a″) *if every substructure of* $\mathfrak{B}$ *of power* $\leqq n$ *is in K, or if* $\mathfrak{B}$ *is a substructure of some* $\mathfrak{A}\in K$, *then* $\mathfrak{B}\in K$;

(b″) *K is the class of all models of some set of sentences of the form*

$$\forall v_0 \cdots \forall v_{n-1}\Phi,$$

where Φ *has no quantifiers.*

In case $n=0$, Theorem A reduces to the following theorem of Łoś [6] and Henkin [3] (which is the dual of the theorem of Tarski and Łoś stated above).

The following two conditions are equivalent:

(a_0) *if* $\mathfrak{B}$ *is an extension of some* $\mathfrak{A}\in K$, *then* $\mathfrak{B}\in K$;

(b_0) *K is the class of all models of some set of existential sentences.*

We now state a result of Łoś and Suszko [7] and of Chang [1].

The following three conditions are equivalent:

(a_ω) *if M is a subset of K and, for all n,* $\mathfrak{B}$ *is an n-union of M, then* $\mathfrak{B}\in K$;

(b_ω) *K is the class of all models of some set of* $\forall\exists$ *sentences*;

(c_ω) *the union of any ascending* ω-*chain of members of K belongs to K.*

Only the equivalence of (b_ω) and (c_ω) were actually stated in [7] and [1], but (a_ω) is the natural analogue of (a_n). It is easily seen, however, that (b_ω) implies (a_ω) and that (a_ω) implies (c_ω); the deep part of the result is that (c_ω) implies (b_ω). See also [2], [9], in connection with the condition (b_n).

We shall distinguish between sets and classes, and shall assume the axiom of choice throughout. We refer to [8] for all of our notation, as well as for general references in the theory of models. We always assume that $\mathfrak{A}$ and $\mathfrak{B}$ are structures of an arbitrary but fixed similarity type, and we denote by κ the power of the set of all symbols of the first order logic corresponding to that similarity type. Thus κ is infinite. As in [8], we also consider structures of the form $(\mathfrak{A}, a_c)_{c\in C}$, where each a_c is an element of $\mathfrak{A}$. $|\mathfrak{A}|$ is the set of all elements of $\mathfrak{A}$. $\mathfrak{A}\equiv\mathfrak{B}$ means that $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent.

In [12], the term "union" was used in a different sense than ours. To clarify the situation, let us consider a nonempty set $M=\{\mathfrak{A}_i: i\in I\}$ of structures $\mathfrak{A}_i=\langle A_i, R_i\rangle$ where each R_i is a binary relation on A_i. The union of M in the sense of [12] is the structure $\langle\bigcup_{i\in I} A_i, \bigcup_{i\in I} R_i\rangle$, which always exists and is unique. For the set M to have a union in our sense it is necessary and sufficient that $R_i\cap A_j^2\subseteq R_j$ for all $i, j\in I$. If M does have a union in our sense, then the union of M in the sense of [12] is a union of M in our sense; in fact, any structure $\langle A, R\rangle$ such that $A=\bigcup_{i\in I} A_i$, $\bigcup_{i\in I} R_i\subseteq R$, and $R\cap\bigcup_{i\in I}(A_i^2-R_i)=0$ is a union of M, and these structures are the only unions of M. For M to have an n-union it is necessary and sufficient that M have a union and each set of at most n elements of $\bigcup_{i\in I} A_i$ be included in some A_j. If $n>0$ and M has an n-union, then $\mathfrak{A}$ is an n-union of M if and only if it is a union of M.

We shall use the following definition from [8].

DEFINITION. *$\mathfrak{A}$ is said to be special if it is of infinite power α and there is a* $\mathrm{cf}(\alpha)$*-complete ideal Q in the field of all subsets of $\mathfrak{A}$ such that*:

(1) *each member of Q has power $<\alpha$*;

(2) *$|\mathfrak{A}|$ is the union of a chain of members of Q*; *and*

(3) *whenever $X\subseteq Y\subseteq|\mathfrak{B}|$, Y has power $<\alpha$, f is a function on X into $|\mathfrak{A}|$ with $fX\in Q$, and $(\mathfrak{A}, fx)_{x\in X}\equiv(\mathfrak{B}, x)_{x\in X}$, then f can be extended to a function g on Y into $|\mathfrak{A}|$ with $gY\in Q$ and $(\mathfrak{A}, gy)_{y\in Y}\equiv(\mathfrak{B}, y)_{y\in Y}$.*

We shall call Q a specializing ideal of $\mathfrak{A}$.

The following result of Morley and Vaught is suggested by the definition. Although it is not stated explicitly in [8], it is closely related to the fact that a special structure is universal (Corollary 2.4(a) of [8]), and that sufficiently large special structures are relation-universal (Theorem 3.6 of [8]); indeed, the method of proof is the same.

LEMMA 1. *Suppose $\mathfrak{A}$ is special and of power α, that $\mathfrak{B}\equiv\mathfrak{A}$, and that C is a subset of $|\mathfrak{B}|$ of power $\leqq\alpha$. Then there is a function f on C into $|\mathfrak{A}|$ such that*

$$(\mathfrak{B}, b)_{b\in C}\equiv(\mathfrak{A}, fb)_{b\in C}.$$

PROOF. We may express C as the union of an ascending chain $\langle C_\beta\rangle_{\beta<\mathrm{cf}(\alpha)}$ of sets C_β of power $<\alpha$, and we may assume further that $C_0=0$ and that, when β is a limit ordinal, $C_\beta=\bigcup_{\gamma<\beta} C_\gamma$. Let Q be a specializing ideal of $\mathfrak{A}$. We wish to prove by transfinite induction that there is an ascending chain $\langle f_\beta\rangle_{\beta<\mathrm{cf}(\alpha)}$ of functions f_β on C_β onto a member D_β of Q such that

(1) $$(\mathfrak{B}, b)_{b\in C_\beta} \equiv (\mathfrak{A}, f_\beta b)_{b\in C_\beta}.$$

Suppose that $\beta<\mathrm{cf}(\alpha)$ and $\langle f_\gamma\rangle_{\gamma<\beta}$ has the desired properties. If $\beta=0$, we let $f_\beta=0$, and (1) holds by hypothesis. If β is a positive limit ordinal, let $f_\beta=\bigcup_{\gamma<\beta} f_\gamma$. Then since Q is $\mathrm{cf}(\alpha)$-complete, f_β is a function on C_β onto a member of Q; it is clear that (1) also holds. If $\beta=\gamma+1$, then since $\mathfrak{A}$ is special, f_γ may be extended to a function f_β on C_β onto a member of Q such that (1) holds. Our induction is complete, and the function $f=\bigcup_{\beta<\mathrm{cf}(\alpha)} f_\beta$ satisfies the conclusion of the lemma.

The following much deeper result of Morley and Vaught is proved in [8] as Theorem 3.5.

Lemma 2. *If $\mathfrak{A}$ is infinite, then there are special structures $\mathfrak{B}$ of arbitrarily large power which are elementarily equivalent to $\mathfrak{A}$.*

We also need the following result of Łoś [6, Theorem 7] and Tarski [12, Theorem 1.6].

Lemma 3. *If every universal sentence which holds in $\mathfrak{B}$ holds in $\mathfrak{A}$, then there is a structure $\mathfrak{B}'\equiv\mathfrak{B}$ such that $\mathfrak{A}$ is isomorphic to a substructure of $\mathfrak{B}'$.*

We shall say that Ψ is an $\forall_n\exists$ *sentence* if it is of the form displayed in condition (b_n) of Theorem A.

Lemma 4. *Suppose that $\mathfrak{B}$ is either a finite structure or a special structure of power $\geqq\kappa$, and that every $\forall_n\exists$ sentence which holds throughout K holds in $\mathfrak{B}$. Then $\mathfrak{B}$ is an n-union of some subset of K.*

Proof. Let $b_0, \cdots, b_{n-1}$ be (not necessarily distinct) elements of $|\mathfrak{B}|$, and let Γ be the set of all universal sentences which hold in the structure $(\mathfrak{B}, b_m)_{m<n}$. It is easily seen that the conjunction of any two members of Γ is logically equivalent to a member of Γ. Thus, if we can show that each member of Γ is consistent with $\mathrm{Th}(K)$, then it will follow that Γ is consistent with $\mathrm{Th}(K)$.

Let $\Phi\in\Gamma$. Then Φ is logically equivalent to a member Φ_1 of Γ in which none of the variables $v_0, \cdots, v_{n-1}$ occur. We may form a universal formula Φ_2 from Φ_1 by replacing the constants corresponding to $b_0, \cdots, b_{n-1}$ by the variables $v_0, \cdots, v_{n-1}$, respectively. (Notice that although the elements $b_0, \cdots, b_{n-1}$ need not be distinct, we have n distinct constants in our formal system corresponding to them.) Then $\exists v_0 \cdots \exists v_{n-1}\Phi_2$ holds in $\mathfrak{B}$. Moreover, $\exists v_0, \cdots, \exists v_{n-1}\Phi_2$ is consistent with $\mathrm{Th}(K)$, because its negation is an $\forall_n\exists$ sentence which does not hold in $\mathfrak{B}$ and so does not belong to $\mathrm{Th}(K)$. It follows that Φ_1 is consistent with $\mathrm{Th}(K)$, and hence Φ is consistent with $\mathrm{Th}(K)$. We conclude that $\Gamma \cup \mathrm{Th}(K)$ is a consistent set of sentences. By the compactness and Löwenheim-Skolem theorems, $\Gamma \cup \mathrm{Th}(K)$ has a model $(\mathfrak{A}, a_m)_{m<n}$ of power at most κ. Since K is an elementary class in the wider sense, we have $\mathfrak{A} \in K$.

We may now apply our previous lemmas. By Lemma 3, there is a structure $(\mathfrak{B}', c_m)_{m<n} \equiv (\mathfrak{B}, b_m)_{m<n}$ such that $(\mathfrak{A}, a_m)_{m<n}$ is isomorphic to a substructure $(\mathfrak{A}', c_m)_{m<n}$ of $(\mathfrak{B}', c_m)_{m<n}$. Suppose first that $\mathfrak{B}$ is special. It is easily seen that since $\mathfrak{B}$ is special, $(\mathfrak{B}, b_m)_{m<n}$ is also special. Furthermore, since the power of $\mathfrak{A}$ is $\leqq \kappa$ and the power of $\mathfrak{B}$ is $\geqq \kappa$, we see that the power of $\mathfrak{B}$ is $\geqq$ the power of $\mathfrak{A}'$. By Lemma **1**, there is a function f on $|\mathfrak{A}'|$ into $|\mathfrak{B}|$ such that

$$(\mathfrak{B}, b_m, fa)_{m<n, a \in |\mathfrak{A}'|} \equiv (\mathfrak{B}', c_m, a)_{m<n, a \in |\mathfrak{A}'|}.$$

The restriction of $\mathfrak{B}$ to the range of f is isomorphic to $\mathfrak{A}'$, and hence to $\mathfrak{A}$. Also, the elements c_m, $m<n$ belong to $\mathfrak{A}'$, and $fc_m = b_m$ for each $m<n$, so $b_0, \cdots, b_{n-1}$ are elements of the range of f. This shows that the elements $b_0, \cdots, b_{n-1}$ are included in a substructure of $\mathfrak{B}$ which is isomorphic to $\mathfrak{A}$ and thus belongs to K.

Let us now consider the other possibility, that $\mathfrak{B}$ is finite. Since (e.g. by Theorem 1.4 of [**12**]) any structure which is elementarily equivalent to a finite structure is isomorphic to it, $(\mathfrak{B}', c_m)_{m<n}$ is isomorphic to the finite structure $(\mathfrak{B}, b_m)_{m<n}$. Then $(\mathfrak{A}, a_m)_{m<n}$ is isomorphic to a substructure $(\mathfrak{A}', b_m)_{m<n}$ of $(\mathfrak{B}, b_m)_{m<n}$. Thus $\mathfrak{A}' \in K$, $\mathfrak{A}'$ is a substructure of $\mathfrak{B}$, and $b_0, \cdots, b_{n-1}$ are elements of $\mathfrak{A}'$.

We have shown in either case that $\mathfrak{B}$ is an n-union of a subset of K, namely of the set of all substructures of $\mathfrak{B}$ which belong to K.

Proof of Theorem A. Assume the condition (a_n). Let Γ be the set of all $\forall_n \exists$ sentences which hold throughout K. Then any member of K is a model of Γ. Let $\mathfrak{B}$ be a model of Γ. By Lemma 2, there is a structure $\mathfrak{B}' \equiv \mathfrak{B}$ which is either finite or a special structure of power $\geqq \kappa$. Now by Lemma 4, $\mathfrak{B}'$ is an n-union of some subset of K. Hence by (a_n), $\mathfrak{B}'$ belongs to K, and therefore $\mathfrak{B} \in K$. We have shown that K is exactly the class of all models of Γ, and so (b_n) holds.

Finally, if (b_n) holds and $\mathfrak{B}$ is an n-union of some subset of K, then it is easily checked that every $\forall_n \exists$ sentence which holds throughout K holds in $\mathfrak{B}$, and thus $\mathfrak{B} \in K$ and (a_n) is true.

We conclude by stating a more general "relativized" form of Theorem A. This type of relativization is in the spirit of A. Robinson's papers [**10**] and [**11**], where relativized forms of some of the earlier results mentioned in our introduction are given; see also [**6**; **12**].

Theorem A*. *Suppose that K, L are elementary classes in the wider* *sense, and that n is a fixed natural number. Then the following two conditions are equivalent*:

(a_n^*) *if* $\mathfrak{B}\in L$ *and* $\mathfrak{B}$ *is an n-union of some subset* M *of* $K\cap L$, *then* $\mathfrak{B}\in K$;

(b_n^*) *there is a set* Γ *of* $\forall_n\exists$ *sentences such that* $K\cap L$ *is the class of all models of* $\Gamma\cup\mathrm{Th}(L)$.

The proof of Theorem A* is an obvious modification of the proof of Theorem A. Lemmas 2 and 4 are used in the same way, and the only change necessary is that at each point in the proof of Theorem A we must consider only structures which belong to L.

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