A METHOD FOR CONSTRUCTING DIRICHLET ALGEBRAS[1]

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I. **Introduction.** We give here some methods for the construction of new Dirichlet algebras out of old ones. The arguments and results are extensions of those we have given in [1]. As one application, we obtain a proper Dirichlet subalgebra of the algebra of functions continuous on the unit circle which extend analytically to the disk, and this answers a question asked in [4].

Related results, arrived at independently, are contained in Glicksberg [3].

We start with a Lemma, perhaps well known, of which a special case was used in [1].

LEMMA. *Let B be a Banach space, B^* its conjugate space. Let U, V be weak-* closed subspaces of B^*. Write W for the vector space sum of U and V. Then W is weak-* closed provided there exists a positive constant k such that*

$$\|u\| + \|v\| \leqq k \cdot \|u + v\|, \qquad \text{all } u \text{ in } U, \quad v \text{ in } V.$$

PROOF. Let $S = \{x \in B^* : \|x\| \leqq 1\}$. According to the Krein-Smulian theorem [2, p. 429], it suffices to show $S \cap W$ is compact (in the weak-* topology). Put $Q = \{u+v : u \in U,\ v \in V,\ \|u\| \leqq 1,\ \|v\| \leqq 1\}$. Evidently Q is compact. Using our hypothesis, one readily checks that $S \cap W = S \cap k \cdot Q$, a compact set.

We shall apply this lemma below. First we introduce some notations. Let X be a compact Hausdorff space, $C(X)$ the Banach space of all complex-valued continuous functions on X. The conjugate space is identified with the space of complex Baire measures on X. For a closed subalgebra A of $C(X)$, we denote by $A^\perp$ the set of all such measures μ satisfying $\int f d\mu = 0$, all $f \in A$. If μ is a measure, $|\mu|$ is the total variation of μ, a positive measure; $\|\mu\|$ is the norm of μ as a linear functional on $C(X)$. If $f \in C(X)$, $f\mu$ is the measure defined by $(f\mu)(E) = \int_E f d\mu$. If ψ is a homeomorphism of X on X, $\mu \circ \psi$ is the measure defined by $(\mu \circ \psi)(E) = \mu(\psi(E))$. If A is a closed subalgebra of $C(X)$ which contains the constants and separates the points of X, we shall call it a *function algebra on X*. If in addition the real parts of functions in A uniformly approximate to all real continuous functions on X, we call A a *Dirichlet algebra on X*. It is easy to see that A is a

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Dirichlet algebra if and only if $A^{\perp}$ contains no nonzero real measures. We shall need the following fact (see [5]): If A is any Dirichlet algebra on X and M is any maximal ideal of A, there exists a unique positive measure σ_M on X with total mass 1, such that

$$\int f d\sigma_M = 0, \qquad \text{all } f \text{ in } M.$$

For a general discussion of Dirichlet algebras, see [5].

If A is any function algebra on a space X, we denote by $S(A)$ the space of maximal ideals of X, taken in the Gelfand topology and hence a compact Hausdorff space. The space X has a natural homeomorphic embedding in $S(A)$ as a closed subset, and we shall regard X as contained in $S(A)$. A may be regarded as a function algebra on $S(A)$.

If A and B are function algebras on X, we shall denote by $S(A) \# S(B)$ the compact space obtained by attaching $S(A)$ to $S(B)$ along X, via the natural embeddings of X.

EXAMPLE. If A is a function algebra on the circle, and $S(A)$ is the closed disk, $S(A) \# S(A)$ is a 2-sphere (see III below).

II. General results.

THEOREM 1. *Let A and B be Dirichlet algebras on X. Suppose there is a Baire set $E \subset X$ such that $|\nu|(E) = |\mu|(X-E) = 0$ for every $\mu \in A^{\perp}$, $\nu \in B^{\perp}$. Then $A \cap B$ is a Dirichlet algebra on X, and $S(A \cap B) = S(A) \# S(B)$.*

PROOF. The hypothesis implies that $\|\mu+\nu\| = \|\mu\| + \|\nu\|$ for every $\mu \in A^{\perp}$, $\nu \in B^{\perp}$. Applying the lemma, we find that $A^{\perp} + B^{\perp}$ is weak-* closed, and hence $(A \cap B)^{\perp} = A^{\perp} + B^{\perp}$. But if $\mu \in A^{\perp}$, $\nu \in B^{\perp}$, and $\mu + \nu$ is real, then μ and ν are each real (since μ and ν are mutually singular), so $\mu = \nu = 0$. Thus $A \cap B$ is a Dirichlet algebra. To prove the second assertion, we define a map from $S(A) \# S(B)$ into $S(A \cap B)$ as follows: If M is a maximal ideal of A or B, $M' = M \cap A \cap B$ is a maximal ideal of $A \cap B$. It is easy to see that the map: $M \to M'$ is continuous. To see that the map: $M \to M'$ is injective, recall that to each maximal ideal M there is associated a unique positive measure σ_M on X, with total mass 1, such that $\int f d\sigma_M = 0$ for all $f \in M$. Evidently, $\sigma_M = \sigma_{M'}$. If $M' = N'$, $\sigma_M - \sigma_N$ is a real measure annihilating $A \cap B$, so $\sigma_M = \sigma_N$. If M and N are both ideals of A (or B), then it follows that $M = N$. It remains to consider the possibility: $M \in S(A)$, $N \in S(B)$. But if $M \in S(A) - X$, $\sigma_M(X-E) = 0$. To prove this, we observe that for any $f \in M$, $f\sigma_M \in A^{\perp}$, and so $\int_{X-E} |f| d\sigma_M = 0$.

For each $x \in X$ there is an $f_x \in M$ with $f_x(x) \neq 0$. Using the continuity of the f_x and the compactness of X, we get $f_1, \cdots, f_k$ in M such that $F = \sum_{i=1}^{k} |f_i| > 0$ on X. But

$$\int_{X-E} F d\sigma_M = \sum_{i=1}^{k} \int_{X-E} |f_i| \, d\sigma_M = 0$$

which implies that $\sigma_M(X-E)=0$. Similarly, if $N \in S(B)-X$, $\sigma_N(E) = 0$. Since $\sigma_M = \sigma_N$, $\sigma_M(X)=0$, which is false. It follows that M and N correspond to the same point of X. Thus the map: $M \to M'$ is injective.

To see that the map is surjective, let L now denote any maximal ideal of $A \cap B$. We must show that L is contained in a maximal ideal of either A or B. Put $\sigma_L = \phi_1 + \phi_2$ where $\phi_1(Z) = \sigma_L(Z \cap E)$ for any Baire set $Z \subset X$. Then ϕ_1 and ϕ_2 are positive measures, not both zero. Suppose $\phi_1 \neq 0$. We then assert that L is contained in a maximal ideal of A. To prove this, it suffices to show that the ideal of A generated by L is proper. Now for any $f \in L$, $f\sigma_L \in (A \cap B)^{\perp}$, so $f\phi_1 + f\phi_2 = \mu + \nu$, where $\mu \in A^{\perp}$ and $\nu \in B^{\perp}$. Since $|f\phi_1 - \mu| (X-E) = 0 = |\nu - f\phi_2| (E)$, we conclude that $f\phi_1 = \mu$. Thus $\int fg d\phi_1$ for every $f \in L$, $g \in A$. Since $\int 1 d\phi_1 > 0$, it follows that the ideal generated by L in A is proper.

Let A be a Dirichlet algebra on X, and let ψ be a homeomorphism of X on itself. We define $A(\psi) = \{f \in A : f \circ \psi \in A\}$. $A(\psi)$ is clearly a closed subalgebra of A containing the constants. It may, of course, reduce to the constants.

We call ψ *singular* (with respect to A) if there exists a Baire set $E \subset X$ such that

$$(*) \qquad |\mu| (X - E) = |\mu| (\psi^{-1}(E)) = 0, \qquad \text{all } \mu \in A^{\perp}.$$

THEOREM 2. *If ψ is singular, $A(\psi)$ is a Dirichlet algebra on X. Moreover, $S(A(\psi)) = S(A) \#_{\psi} S(A)$, the space obtained by attaching $S(A)$ to $S(A)$ along X via the map ψ; and $A(\psi)$ is a proper subalgebra of A, unless $A = C(X)$.*

PROOF. Let $B = \{f \in C(X) : f \circ \psi \in A\}$. Since A is a Dirichlet algebra on X, so is B, and clearly $A(\psi) = A \cap B$. Now $\nu \in B^{\perp}$ if and only if $\int f \circ \psi^{-1} d\nu = 0$ for all $f \in A$, and so if and only if $\nu \circ \psi \in A^{\perp}$. Let E be the set satisfying (*). Then $|\mu| (X-E) = 0$ for all $\mu \in A^{\perp}$; also if $\nu \in B^{\perp}$, $|\nu| (E) = |\nu \circ \psi| (\psi^{-1}(E)) = 0$, since $\nu \circ \psi \in A^{\perp}$. Thus Theorem 1 applies to the algebras A and B, yielding that $A(\psi) = A \cap B$ is a Dirichlet algebra on X, and that $S(A(\psi)) = S(A) \# S(B) = S(A) \#_{\psi} S(A)$. If $A(\psi) = A$, then $A \subset B$, so $A^{\perp} \supset B^{\perp}$ so for every $\nu \in B^{\perp}$, $|\nu| (X-E) = |\nu| (E) = 0$, thus $\nu = 0$, and hence $B = C(X)$, whence $A = C(X)$.

Suppose next that A is a function algebra on X and G a group of homeomorphisms of X on itself, such that $f \circ g \in A$, for every $f \in A$,

$g \in G$. In a natural way each $g \in G$ extends to a homeomorphism of $S(A)$ on itself: if M is a maximal ideal of A, $g(M) = \{f \in A \mid f \circ g \in M\}$. Put

$$A' = \{f \in A : f = f \circ g \text{ for every } g \in G\}.$$

Then A' is a uniformly closed algebra of functions on X. We denote by X/G the identification space induced by G. A' may be regarded as an algebra of functions on X/G.

THEOREM 3. *Suppose G is finite. Then $S(A') = S(A)/G$. If A is a Dirichlet algebra on X, then A' is a Dirichlet algebra on X/G.*

PROOF. We may regard A' as an algebra of functions on $S(A)/G$. Thus there is an obvious continuous map τ from $S(A)/G$ into $S(A')$.

To see that τ is injective, let m_1, m_2 be points of $S(A)$ giving rise to distinct points of $S(A)/G$, i.e., such that $g(m_1) \neq m_2$ for all $g \in G$. Choose $f \in A$ such that $f(g(m_1)) \neq 0$ for all $g \in G$, $f(m_2) = 0$. Put $F = \prod_{g \in G} f \circ g$. Then $F \in A$, $F \circ g = F$ for every $g \in G$, so $F \in A'$, and $F(m_1) \neq 0 = F(m_2)$. Thus $\tau(m_1)$, $\tau(m_2)$ are distinct elements of $S(A')$.

To show that τ is surjective, consider a maximal ideal M of A' and let M_1 denote the ideal generated by M in A. Suppose M_1 is not proper. Then we can find $f_1, \cdots, f_n$ in M and $k_1, \cdots, k_n$ in A with

$$\sum_{i=1}^{n} f_i k_i = 1.$$

Put $k_i' = \sum_{g \in G} k_i \circ g$. Then $k_i' \in A'$ and

$$\sum_{i=1}^{n} f_i k_i' = N = \text{order of } G.$$

Hence M is the unit ideal in A', a contradiction. Thus M_1 is proper, and τ maps the point of $S(A)/G$ induced by M_1 on M. Thus τ is surjective, and so $S(A)/G = S(A')$.

We now let A be a Dirichlet algebra on X. Let u be a real continuous function on X such that $u \circ g = u$ for all $g \in G$, so that u can be regarded as defined on X/G. Let $\epsilon > 0$ be arbitrary. Since A is a Dirchlet algebra, there exists f in A such that $\|\operatorname{Re} f - u\| < \epsilon$, (where $\| \ \|$ denotes the maximum modulus on X).

Put $F = (1/N) \sum_{g \in G} f \circ g$. Clearly $F \in A'$ and $\|\operatorname{Re} F - u\| < \epsilon$, since $\|\operatorname{Re} f \circ g - u\| < \epsilon$ for each $g \in G$. Hence A' is a Dirichlet algebra on X/G.

NOTE. In the latter part of the theorem, the same argument can be made if G is compact (instead of finite), with integration over G replacing summation. Also, the same method shows that if A is a maximal subalgebra of $C(X)$, then A' is a maximal subalgebra of $C(X/G)$.

III. **Applications.** Let Γ denote the unit circle $|z|=1$ in the z-plane, and A_0 the algebra of all continuous functions on Γ which admit continuous extensions to $|z|\leqq 1$, analytic on $|z|<1$. It is well known that: A_0 is a Dirichlet algebra on Γ; $S(A_0)$ can be identified with the closed disk $|z|\leqq 1$; every measure in $A_0^\perp$ is absolutely continuous with respect to Lebesgue measure on Γ (F. and M. Riesz); A_0 is a maximal subalgebra of $C(\Gamma)$ (see [5]).

Let ψ be a homeomorphism of Γ on Γ such that for some Borel set E of Lebesgue measure 2π, $\psi^{-1}(E)$ has Lebesgue measure zero. We shall call ψ *singular.* Recall that by definition, $A_0(\psi)=\{f\in A_0 \mid f\circ\psi \in A_0\}$. In view of the facts summarized above, Theorem 2 applies, to give:

Corollary 1. *$A_0(\psi)$ is a Dirichlet algebra on Γ, and is a proper subalgebra of A_0. $S(A_0(\psi))$ is a 2-sphere.*

Let ψ be a singular homeomorphism of Γ on itself such that $\psi\circ\psi$ $=$identity. Clearly $f\circ\psi\in A_0(\psi)$ for every $f\in A_0(\psi)$. Put

$$A_\psi = \{f\in A_0 \mid f = f\circ\psi\}.$$

Let G be the two element group generated by ψ. Then $A_\psi=\{f\in A_0(\psi) \mid f = f\circ g \text{ for every } g \text{ in } G\}$. Thus Theorem 3 can be applied, with $A=A_0(\psi)$ and $A'=A_\psi$. We get, first

Corollary 2. *A_ψ is a Dirichlet algebra on Γ/ψ.*

The topology of $S(A_\psi)$ and of Γ/ψ depends on whether or not ψ preserves orientation on Γ.

Corollary 3. *If ψ reverses orientation, Γ/ψ is (homeomorphic to) a closed interval and $S(A_\psi)$ is a 2-sphere. If ψ preserves orientation and has no fixed points, Γ/ψ is a circle and $S(A_\psi)$ is a (real) projective plane.*

Proof. The assertions for Γ/ψ are easily verified. By Theorem 3, $S(A_\psi)=S(A_0(\psi))/G$. Since $S(A_0(\psi))$ is a 2-sphere, $S(A_0(\psi))/G$ is easily seen to be a closed disk with the boundary identification induced by ψ. From this the assertions follow.

Note. The situation when ψ reverses orientation was described in [1]. The method of proof of the following result was used in the corresponding theorem in [1].

Theorem 4. *A_ψ is a maximal subalgebra of $C(\Gamma/\psi)$.*

PROOF. Let B be a closed subalgebra of $C(\Gamma/\psi)$. We may regard B as a closed subalgebra of $C(\Gamma)$ such that $f=f\circ\psi$ for every $f\in B$. Assume $A_\psi\subset B$. Let $\mathfrak{O}$ denote the space of all measures ν on Γ such that $\nu=-\nu\circ\psi$, so $\mathfrak{O}=C(\Gamma/\psi)^\perp$ under the obvious identifications. If $\mu\in A_0^\perp$ and $\nu\in\mathfrak{O}$, we have

$$\begin{aligned}\|\mu\| &= \frac{1}{2}\|\mu+\mu\circ\psi\| = \frac{1}{2}\|\mu+\nu+\mu\circ\psi+\nu\circ\psi\| \\ &\leqq \frac{1}{2}(\|\mu+\nu\|+\|\mu\circ\psi+\nu\circ\psi\|) = \|\mu+\nu\|.\end{aligned}$$

Hence by the Lemma $A_0^\perp+\mathfrak{O}$ is weak-* closed, and therefore $=A_\psi^\perp$. Thus if $\lambda\in B^\perp$, $\lambda=\mu+\nu$ for some $\mu\in A_0^\perp$, $\nu\in\mathfrak{O}$. If $B\neq C(\Gamma/\psi)$, then we have such a λ outside $\mathfrak{O}$, so $0\neq\mu=\lambda-\nu\in A_0^\perp\cap B^\perp$. Now for any $f\in B$, $f\mu\in B^\perp$, so $f\mu=\mu_1+\nu_1$ for some $\mu_1\in A_0^\perp$, $\nu_1\in\mathfrak{O}$. Thus $\mu_1-f\mu=f\mu\circ\psi-\mu_1\circ\psi$. Since ψ is singular, $f\mu-\mu_1=0$, or $f\mu\in A_0^\perp$. Thus $\int gfd\mu=0$ for every $g\in A_0, f\in B$. Since $\mu\neq 0$ and A_0 is a maximal algebra, this implies $B\subset A_0$, and thus $B=A_\psi$. Hence A_ψ is maximal.

APPENDIX. As an application of the algebras $A_0(\Psi)$ introduced above, we now give a closure result on the unit circle Γ. Let Ψ be a homeomorphism of Γ which reverses orientation on Γ. We do ***not*** assume here that Ψ is singular.

THEOREM 5. *Every continuous function on Γ can be uniformly approximated by linear combination of powers z^n, $n\geqq 0$, and Ψ^n, $n\geqq 0$.*

The proof makes use of arguments given in [1]. Let $f\in A_0(\Psi)$. Let α be a value taken by f in $|z|<1$. Suppose $\alpha\notin f(\Gamma)$. Put $g=f-\alpha$. Then $g\in A_0(\Psi)$ and $\mathrm{var}_\Gamma \arg g>0$. But $\mathrm{var}_\Gamma\arg g=-\mathrm{var}_\Gamma\arg g(\Psi)$, since Ψ reverses direction, and var arg $g(\Psi)\geqq 0$, since $g(\Psi)\in A_0$. This is a contradiction, and so $\alpha\in f(\Gamma)$. Thus $f(|z|<1)\subseteq f(\Gamma)$. We conclude that, unless f is a constant, $f(\Gamma)$ has positive area in the plane.

Let μ be any measure on Γ with $\mu\perp z^n$, $n\geqq 0$, and $\mu\perp\Psi^n$, $n\geqq 0$. All we need to do is to show that μ must be 0. By the F. and M. Riesz theorem, $d\mu=h(t)dt$ on $(-\pi,\pi)$, where there is some η in the Hardy class H^1 with $\eta(0)=0$ and $\eta(e^{it})=h(t)$. Put $H(\theta)=\int_{-\pi}^{\theta}h(t)dt$. It is easy to verify that $H\in A_0$.

Because we may rotate, it is no loss of generality to assume that $\Psi(-1)=-1$. We can then set: $\Psi(e^{it})=e^{i\psi(t)}$, where ψ is a strictly decreasing continuous function on $(-\pi,\pi)$ with $\psi(-\pi)=\pi$, $\psi(\pi)=-\pi$. Now

$$\int_{-\pi}^{\pi} h(t)e^{in\psi(t)}dt = 0, \qquad n\geqq 0.$$

Integrating by parts, we get

$$\int_{-\pi}^{\pi} H(t)d(e^{in\psi(t)}) = 0, \qquad n \geqq 0.$$

Putting $u=\psi(t)$ in this integral, we have

$$\int_{-\pi}^{\pi} H(\psi^{-1}(u))d(e^{inu}) = 0, \qquad n \geqq 0.$$

Hence $H(\psi^{-1})\in A_0$. Thus $H\in A_0(\Psi^{-1})$. Since Ψ^{-1} also reverses orientation, we conclude by the above that either H is constant or $H(\Gamma)$ has positive area. But H is absolutely continuous. Hence H is constant and so 0, whence $\mu=0$, and we are done.

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