EXTENSION OF C^∞ FUNCTIONS DEFINED IN A HALF SPACE

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The following question arises in connection with manifolds with boundary: given a function f defined and C^∞ in a half space, and all of whose derivatives have continuous limits at the boundary, can f be extended to a C^∞ function in the whole space? More specifically, let $x \in \boldsymbol{R}^n$, $t \in \boldsymbol{R}^1$, $S_+ = \boldsymbol{R}^n \times \{t > 0\}$, and $D_+ = \{f: f$ in $C^\infty(S^+)$, f and all its derivatives have continuous limits as $t \to 0+\}$. D_+ has the topology of uniform convergence of each derivative on compact subsets of the closure of S_+ in $\boldsymbol{R}^{n+1}$, and $C^\infty(\boldsymbol{R}^{n+1})$ has a corresponding topology.

THEOREM. *There is a continuous linear extension operator* $E: D_+ \to C^\infty(\boldsymbol{R}^{n+1})$ *such that* $Ef(x, t) = f(x, t)$ *for* $t > 0$.

The theorem is an easy consequence of the following:

LEMMA. *There are sequences* $\{a_k\}$, $\{b_k\}$ *such that* (i) $b_k < 0$, (ii) $\sum |a_k| \, |b_k|^n < \infty$ *for* $n = 0, 1, 2, \cdots$, (iii) $\sum_0^\infty a_k (b_k)^n = 1$ *for* $n = 0, 1, 2, \cdots$, *and* (iv) $b_k \to -\infty$.

Granted the lemma, the extension can be made just as for C^n functions, by a combination of reflections. Let ϕ be a C^∞ function on $\boldsymbol{R}^1$ with $\phi(t) = 1$ for $0 \leqq t \leqq 1$, $\phi(t) = 0$ for $t \geqq 2$. For $t < 0$ define $(Ef)(x, t) = \sum_0^\infty a_k \phi(b_k t) f(x, b_k t)$. Then because $b_k \to -\infty$, the sum is finite for each $t < 0$; because $\sum |a_k| \, |b_k|^n < \infty$, all derivatives of Ef converge as $t \to 0-$, and uniformly in each bounded set $\{|x| \leqq R\}$; because $\sum_0^\infty a_k (b_k)^n = 1$, these limits agree with those for $t \to 0+$. Thus if $(Ef)(x, t) = f(x, t)$ for $t > 0$ and $(Ef)(x, 0) = \lim_{t \to 0} f(x, t)$, Ef is in $C^\infty(\boldsymbol{R}^{n+1})$. The continuity of $E: D_+ \to C^\infty(\boldsymbol{R}^{n+1})$ follows from $\sum |a_k| \, |b_k|^n < \infty$.

The lemma is also easy. Set $b_k = -2^k$. Then the solutions a_{kN} of $\sum_0^N X_k (b_k)^n = 1$ $(n = 0, \cdots, N)$ are, by Cramer's rule and Vandermonde's expansion, $a_{kN} = A_k B_{kN}$, where $A_k = \prod_{j=0}^{k-1} (1 + 2^j)/(2^j - 2^k)$, and $B_{kN} = \prod_{j=k+1}^N (1 + 2^j)/(2^j - 2^k)$. Then $|A_k| \leqq \prod_0^{k-1} 2^{j+2-k} = 2^{-(k^2-3k)/2}$, and $\log (B_{kN}) = \sum_{k+1}^N \log (1 + (1 + 2^k)/(2^j - 2^k)) < \sum (1 + 2^k)/(2^j - 2^k) < 4$.

Thus B_{kN} is monotone increasing with N, and converges to $B_k \leqq e^4$. Setting $a_k = A_k B_k$, we have $|a_k| < e^4 2^{-(k^2-3k)/2}$, so $\sum |a_k| \, |b_k|^n < \infty$

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for each n. Setting $a_{kN}=A_kB_{kN}$ for $k\leqq N$, and $a_{kN}=0$ for $N\leqq k$, we have $\sum_0^\infty a_{kN}(b_k)^n=1$ for $N\geqq n$. Since $|a_{kN}|\ |b_k|^n\leqq|a_k|\ |b_k|^n$ and $\sum|a_k|\ |b_k|^n<\infty$, we have finally $\sum_0^\infty a_k(b_k)^n=\lim\sum a_{kN}(b_k)^n=1$.

The author is indebted to the referee for the remark that reflections were used to extend C^1 functions by L. Lichtenstein [2] in 1929, and for C^m functions by M. R. Hestenes [1] in 1941. There is also a rather difficult result on extension of C^∞ functions, due originally to Whitney [3] and proved also in [1], but this is different from our extension in two important respects. Whitney extends C^∞ functions from an arbitrary closed set; but the extension is not linear, and it is not shown to be continuous in any functional topology. The extension given above is continuous in many of the functional topologies currently in use. The principal example, other than C^∞, is that of the Soboleff spaces $W_p^k(D)$ $(1\leqq p\leqq\infty,\ k=0,\ 1,\ 2,\ \cdots)$ consisting of functions in $L^p(D)$ whose derivatives of order $\leqq k$ are in $L^p(D)$. Consequently the extension is also continuous in the inverse limit topology of $W_p^\infty(D)=\bigcap W_p^k(D)$. It can also be used to extend a differentiable function from a half Banach space, if the function and all its derivatives are bounded on bounded sets.

Bibliography

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