ON AN INEQUALITY OF OPIAL AND BEESACK

N. LEVINSON[1]

In further simplifying the proof of an inequality of Opial [3], which had already been simplified by Olech [2], Beesack [1] proved the inequality

$$\int_0^a |y(x)y'(x)|\,dx \leqq \frac{1}{2}a\int_0^a (y'(x))^2dx$$

for real $y(x)$ absolutely continuous on $(0, a)$ and with $y(0)=0$. Equality holds only for $y=bx$ where b is a constant.

Here an even simpler proof will be given. Moreover one can take $y(x)$ as complex valued. Clearly

$$\begin{aligned}\int_0^a |y'(x)y(x)|\,dx \\ = \int_0^a |x^{1/2}y'(x)| \left| x^{-1/2}\int_0^x y'(\xi)d\xi \right| dx \leqq (AB)^{1/2},\end{aligned} \tag{1}$$

where

$$A = \int_0^a x\,|y'(x)|^2dx \tag{2}$$

and

$$B = \int_0^a x^{-1}\left|\int_0^x y'(\xi)d\xi\right|^2 dx$$

by the Schwarz inequality. By the same inequality

$$\left|\int_0^x y'(\xi)d\xi\right|^2 \leqq x\int_0^x |y'(\xi)|^2d\xi$$

with equality only if $y'=b$, a constant, almost everywhere. Hence

$$B \leqq \int_0^a \left(\int_0^x |y'(\xi)|^2d\xi\right) dx,$$

Received by the editors April 15, 1963.

[1] Supported in part by the Office of Naval Research and the National Science Foundation, NSF GP-149.

and therefore, inverting the order of integration, we get

$$B \leqq \int_0^a |y'(x)|^2(a-x)dx. \tag{3}$$

Since

$$(AB)^{1/2} \leqq \frac{1}{2}(A+B),$$

(1), (2) and (3) give

$$\int_0^a |y(x)y'(x)|\,dx \leqq \frac{1}{2}a\int_0^a |y'(x)|^2dx, \tag{4}$$

which completes the proof.

It has already been seen that inequality occurs unless $y'(x)=b$ almost everywhere. This with $y(0)=0$ implies

$$y(x) = bx.$$

In this case equality occurs in (4).

References

1. Paul R. Beesack, *On an integral inequality of Z. Opial*, Trans. Amer. Math. Soc. **104** (1962), 470–475.

2. C. Olech, *A simple proof of a certain result of Z. Opial*, Ann. Polon. Math. **8** (1960), 61–63.

3. Z. Opial, *Sur une inégalité*, Ann. Polon. Math. **8** (1960), 29–32.

Massachusetts Institute of Technology