ON THE SUMS $\sum_{k=0}^n k^p$ AND $\sum_{k=0}^n (-1)^k k^p$

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1. Introduction. Let u_k satisfy a linear, difference equation of order m with real, constant coefficients:

$$(1.1) \quad u_{m+k} + a_1 u_{m+k-1} + a_2 u_{m+k-2} + \cdots + a_m u_k = 0, \\ a_0 = 1, a_m \neq 0, k = 0, 1, \dots$$

Using our formula, (2.1), for $\sum_{k=0}^n u_k r^k$, we will give a closed form for $\sum_{k=0}^\infty u_k r^k$, as well as closed forms for $\sum_{k=0}^n k^p$ and $\sum_{k=0}^n (-1)^k k^p$, which are also expressible in closed form by means of Bernoulli and Euler polynomials of the first order, respectively. We will show that

$$(1.2) \quad -2^{p+1} \sum_{k=0}^n (-1)^k k^p \\ = \sum_{k=0}^p \sum_{s=0}^{p-k} (-1)^s \binom{p+1}{s+k+1} [(-1)^{n+1} (n+1+s)^p - s^p] \\ (p = 0, 1, \dots),$$

$$(1.3) \quad \sum_{k=0}^n k^p = \sum_{k=0}^p (-1)^k \binom{n+p+1-k}{p+1} \sum_{s=0}^{p-k} (-1)^s \binom{p+1}{s+k+1} (n+1+s)^p \\ (p = 0, 1, \dots).$$

Closed forms for $\sum_{k=0}^n k^p u_k r^k$, $\sum_{k=0}^n u_k r^k \cos(k\theta)$, and $\sum_{k=0}^n u_k r^k \sin(k\theta)$ may also be obtained from (2.1), but the details are omitted.

2. A lemma. We now derive the following result:

LEMMA 1. Let u_k satisfy (1.1). Then

$$(2.1) \quad - \left[1 + \sum_{i=1}^m a_i r^i \right] \sum_{k=0}^n u_k r^k = \sum_{k=0}^{m-1} \left[\sum_{j=0}^{m-1-k} a_j u_{n+m-k-j} \right] r^{n+m-k} \\ - \sum_{k=0}^{m-1} \left[\sum_{j=0}^{m-1-k} a_j u_{m-1-k-j} \right] r^{m-1-k}.$$

PROOF OF (2.1). Let A_i , $i = 1, 2, \dots, m$, be real constants, and let

$$(2.2) \quad v_{m+k} + A_1 v_{m+k-1} + A_2 v_{m+k-2} + \cdots + A_m v_k = 0, \quad A_m \neq 0.$$

Defining $\Delta v_k = v_{k+1} - v_k$, (2.2) can be rewritten as

Received by the editors June 1962; and, in revised form, April 15, 1963.

$$(2.3) \quad \Delta v_{m+k-1} + \sum_{j=0}^{m-2} \left[1 + \sum_{i=1}^{j+1} A_i \right] \Delta v_{m+k-2-j} + \left[1 + \sum_{i=1}^m A_i \right] v_k = 0;$$

moreover, by summing both sides of (2.3) for $k=0, 1, \dots, n$, we obtain

$$(2.4) \quad (v_{n+m} - v_{m-1}) + \sum_{j=0}^{m-2} \left[1 + \sum_{i=1}^{j+1} A_i \right] (v_{n+m-1-j} - v_{m-2-j}) \\ + \left[1 + \sum_{i=1}^m A_i \right] \sum_{k=0}^n v_k = 0.$$

If we set $v_k \equiv u_k r^k$, then $A_i = a_i r^i$, $i=1, 2, \dots, m$. Since

$$(2.5) \quad \sum_{j=0}^{m-2} \sum_{i=1}^{j+1} a_i r^{n+m-1-j+i} u_{n+m-1-j} \\ = \sum_{k=0}^{m-2} \sum_{j=1}^{m-1-k} a_j u_{n+m-k-j} r^{n+m-k} \\ = \sum_{k=0}^{m-1} \left[\sum_{j=0}^{m-1-k} a_j u_{n+m-k-j} \right] r^{n+m-k} - \sum_{k=0}^{m-1} u_{n+m-k} r^{n+m-k},$$

(2.4) simplifies to (2.1).

3. Application of Lemma 1. The generating function of u_k is readily obtained from (2.1). Let $R > 0$ and suppose $\sum_{k=0}^{\infty} u_k r^k$ converges for $|r| < R$. Then, for $|r| < R$, $u_n r^n \rightarrow 0$ as $n \rightarrow \infty$. Thus, for $|r| < R$, (2.1) gives, as $n \rightarrow \infty$,

$$(3.1) \quad \left[1 + \sum_{i=1}^m a_i r^i \right] \sum_{k=0}^{\infty} u_k r^k = \sum_{k=0}^{m-1} \left[\sum_{j=0}^{m-1-k} a_j u_{m-1-k-j} \right] r^{m-1-k}.$$

Another proof of (3.1) (based on the additive property of generating functions), is given in Jordan [1, p. 27]. We note that the generating function of u_k is a rational function in r .

Recalling the well-known identity

$$(3.2) \quad \sum_{j=0}^{p+1} (-1)^j \binom{p+1}{j} j^s = \begin{cases} 0 & (s = 0, 1, \dots, p) \\ (p+1)!(-1)^{p+1} & (s = p+1) \end{cases},$$

it follows that $u_k \equiv k^p$, $p=0, 1, \dots$, satisfies

$$\sum_{j=0}^{p+1} (-1)^j \binom{p+1}{j} u_{p+1+k-j} = 0.$$

Thus, from (2.1), with $u_k \equiv k^p$, $m=p+1$, and

$$a_j = (-1)^j \binom{p+1}{j},$$

$j=0, 1, \dots, p+1$, we obtain

$$\begin{aligned} & (-1)^p (r-1)^{p+1} \sum_{k=0}^n k^p r^k \\ (3.3) \quad &= \sum_{k=0}^p \left[\sum_{j=0}^{p-k} (-1)^j \binom{p+1}{j} (n+p+1-k-j)^p \right] r^{n+p+1-k} \\ & \quad - \sum_{k=0}^p \left[\sum_{j=0}^{p-k} (-1)^j \binom{p+1}{j} (p-k-j)^p \right] r^{p-k}. \end{aligned}$$

If we set $r = -1$ in (3.3), we obtain (1.2).

Let $f(r)$ denote the right-hand side of (3.3) so that

$$(-1)^p (r-1)^{p+1} \sum_{k=0}^n k^p r^k = f(r).$$

In §4, it will be shown that $[f^{(s)}(r)]_{r=1} = 0$ for $s=0, 1, \dots, p$. Thus, applying L'Hospital's rule $(p+1)$ times to $f(r)/(r-1)^{p+1}$ for $r=1$, we obtain (1.3) from (3.3), noting that

$$(n+p+1-k)^{(p+1)} = (p+1)! \binom{n+p+1-k}{p+1},$$

where $x^{(p+1)} \equiv x(x-1) \cdots (x-p)$.

REMARKS. Nielsen [2, p. 27, (8)] defines for non-negative integers M, N , and P

$$(3.4) \quad B_P^{M,N}(x) = \sum_{j=0}^P (-1)^j \binom{M+1}{j} (x+P-j)^N.$$

Since

$$\begin{aligned} & \sum_{s=0}^{p-k} (-1)^s \binom{p+1}{s+k+1} (n+1+s)^p \\ &= (-1)^{p-k} \sum_{j=0}^{p-k} (-1)^j \binom{p+1}{j} (n+1+p-k-j)^p, \end{aligned}$$

(1.3) can be rewritten as

$$(3.5) \quad \sum_{k=1}^n k^p = (-1)^p \sum_{s=0}^p \binom{n+1+s}{p+1} B_s^{p,p}(n+1) \quad (p=1, 2, \dots).$$

Noting that $B_{p+1}^{p,p}(x) \equiv 0$ (see (3.2)), we obtain from [2, p. 237, (5), with n and p interchanged, $x=n+1$, $m=p$]

$$(3.6) \quad \sum_{k=1}^n k^p = \sum_{s=0}^p \left[\binom{2n+1+s}{p+1} - \binom{n+1+s}{p+1} \right] B_s^{p,p}(n+1) \\ (p = 1, 2, \dots).$$

Combining (3.5) and (3.6), we obtain

$$(3.7) \quad [1 + (-1)^p] \sum_{k=1}^n k^p = \sum_{s=0}^p \binom{2n+1+s}{p+1} B_s^{p,p}(n+1) \\ (p = 1, 2, \dots).$$

Thus, from (3.7), we conclude that

$$(3.8) \quad \sum_{s=0}^p \binom{2n+1+s}{p+1} B_s^{p,p}(n+1) \equiv 0 \quad (p = 1, 3, 5, \dots),$$

$$(3.9) \quad 2 \sum_{k=1}^n k^p = \sum_{s=0}^p \binom{2n+1+s}{p+1} B_s^{p,p}(n+1) \quad (p = 2, 4, 6, \dots).$$

4. Additional proofs. We will require the following result:

LEMMA 2. For $s=0, 1, \dots, p$, and arbitrary x , we have

$$(4.1) \quad \sum_{i=0}^p \binom{p}{i} x^i \sum_{k=0}^p \binom{x+p-k}{s} \sum_{j=0}^{p-k} (-1)^j \binom{p+1}{j} (p-k-j)^{p-i} \\ \equiv \Delta^{p-s} 1^p,$$

where

$$(4.2) \quad \Delta^m 1^p = \sum_{r=0}^m (-1)^r \binom{m}{r} (m-r+1)^p.$$

PROOF OF (4.1). Let $U(x, s)$ denote the left-hand side of (4.1). Since

$$\sum_{s=0}^p \binom{x}{p-s} \binom{x+p-k}{s} = \binom{2x+p-k}{p},$$

we have

$$(4.3) \quad \sum_{s=0}^p \binom{x}{p-s} U(x, s) \\ = \sum_{i=0}^p \binom{p}{i} x^i \sum_{k=0}^p \binom{2x+p-k}{p} \sum_{j=0}^{p-k} (-1)^j \binom{p+1}{j} (p-k-j)^{p-i} \\ = \sum_{i=0}^p \binom{p}{i} x^i \sum_{k=0}^p \binom{2x+k}{p} \sum_{j=0}^k (-1)^j \binom{p+1}{j} (k-j)^{p-i}.$$

Nielsen [2, p. 28, (10) and (14)] has shown that

$$(4.4) \quad (-1)^{m-n} y^n = \sum_{k=0}^m \binom{y+k-1}{m} \sum_{j=0}^k (-1)^j \binom{m+1}{j} (k-j)^n \quad (m \geq n).$$

Setting $m=p$, $n=p-i$, and $y=2x+1$ in (4.4), we obtain from (4.3)

$$(4.5) \quad \sum_{s=0}^p \binom{x}{p-s} U(x, s) = \sum_{i=0}^p \binom{p}{i} (-x)^i (2x+1)^{p-i} = (x+1)^p.$$

Since the expansion,

$$(x+1)^p = \sum_{s=0}^p \binom{x}{s} \Delta^s 1^p,$$

known as Newton's interpolation formula, is unique (see [3, p. 27], [4, p. 17]), it follows that $U(x, p-s) \equiv \Delta^s 1^p$, and thus $U(x, s) \equiv \Delta^{p-s} 1^p$. This completes the proof of (4.1).

In §3, we claimed that $[f^{(s)}(x)]_{x=1} \equiv f^{(s)}(1) = 0$ for $s=0, 1, \dots, p$. To show this, let $f(x) \equiv h(x) - g(x)$, where (see (3.3))

$$(4.6) \quad h(x) = \sum_{k=0}^p \left[\sum_{j=0}^{p-k} (-1)^j \binom{p+1}{j} (n+p+1-k-j)^p \right] x^{n+p+1-k},$$

$$(4.7) \quad g(x) = \sum_{k=0}^p \left[\sum_{j=0}^{p-k} (-1)^j \binom{p+1}{j} (p-k-j)^p \right] x^{p-k}.$$

We now proceed to show that $h^{(s)}(1) \equiv g^{(s)}(1) = s! \Delta^{p-s} 1^p$, $s=0, 1, \dots, p$. Using the binomial theorem and noting that

$$(n+p+1-k)^{(s)} = s! \binom{n+p+1-k}{s},$$

we obtain from (4.6)

$$(4.8) \quad \begin{aligned} h^{(s)}(1) &= \sum_{k=0}^p (n+p+1-k)^{(s)} \sum_{j=0}^{p-k} (-1)^j \binom{p+1}{j} (n+p+1-k-j)^p \\ &= s! \sum_{i=0}^p \binom{p}{i} (n+1)^i \sum_{k=0}^p \binom{n+p+1-k}{s} \\ &\quad \cdot \sum_{j=0}^{p-k} (-1)^j \binom{p+1}{j} (p-k-j)^{p-i} = s! \Delta^{p-s} 1^p, \end{aligned}$$

having applied (4.1) with $x=n+1$.

If we set $x=0$ in (4.1), the only nonzero term occurs for $i=0$. Thus,

$$(4.9) \quad \sum_{k=0}^p \binom{p-k}{s} \sum_{j=0}^{p-k} (-1)^j \binom{p+1}{j} (p-k-j)^p \equiv \Delta^{p-s} 1^p$$

$$(s = 0, 1, \dots, p);$$

and from (4.7), we have

$$(4.10) \quad g^{(s)}(1) = s! \sum_{k=0}^p \binom{p-k}{s} \sum_{j=0}^{p-k} (-1)^j \binom{p+1}{j} (p-k-j)^p$$

$$= s! \Delta^{p-s} 1^p.$$

Thus, $f^{(s)}(1) \equiv h^{(s)}(1) - g^{(s)}(1) = 0$, $s=0, 1, \dots, p$, as claimed.

Acknowledgment. I wish to thank Professor L. Carlitz for providing me with the outline of the proof for (4.1).

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