# ONE-SIDED INVARIANT SUBSPACES AND DOMAINS OF UNIQUENESS FOR HYPERBOLIC EQUATIONS[1]

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1. **Introduction.** Suppose $t \rightarrow U(t)$ is a continuous one-parameter group of unitary operators on a complex Hilbert space $K$, and let $H$ be the self-adjoint generator of this group. Do there exist real or complex linear closed subspaces of $K$ which are invariant under the semi-group $[U(t) \mid t \geqq 0]$ but which are not invariant under the full group? In §2 we investigate this question under the additional hypothesis that $H$ be a positive-definite operator. Our basic result, when $H \geqq cI > 0$, is that there are no proper one-sided invariant manifolds; the invariant subspaces (real or complex linear) for $[U(t) \mid -\infty < t < \infty]$ are precisely those for $[U(t) \mid t_0 < t < \infty]$, $t_0$ being an arbitrary real number. This fact is exploited in §3 to obtain some sharp results on domains of uniqueness for normalizable (finite-energy) solutions of the Klein-Gordon and related hyperbolic partial differential equations. The general result is that such solutions are uniquely determined by their values on an open time-like backward cone in space-time. This result carries over to the quantized Klein-Gordon field (see Segal [**5**]), and it follows that the collection of field operators $R(f)$, with $f$ a testing function supported on an open time-like backward cone, is complete, i.e. bounded functions of these operators are weakly dense in the space of all bounded operators on the field state space.

Lax, Morawetz and Phillips [**3**] have recently considered scattering for the wave equation, which is the limiting case of the Klein-Gordon equation when the mass $m \rightarrow 0$. An interesting result (cf. our Theorem 3.1) of their investigation is that a finite-energy solution of the wave equation which vanishes in both the forward and backward light cones is zero identically.

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2. **Groups with positive generators.** Suppose $t \to U(t)$ is a one-parameter unitary group on a complex Hilbert space $K$. By Stone's theorem [4] there exists a self-adjoint operator $H$ on $K$ with spectral family $[E_\lambda]$ such that $U(t) = \exp iHt$.

THEOREM 2.1. *If $H$ is a strictly positive operator, then any closed real-linear manifold $M$ in $K$ which is invariant under $[U(t) \mid t \geqq t_0]$ is also invariant under $[U(t) \mid t < t_0]$.*

PROOF. As a preliminary remark, we observe that if $(u, v)$ is the complex inner product on $K$, then $K$ becomes a real Hilbert space $K_r$ under the inner product $[u, v] = \operatorname{Re}(u, v) = [(u, v) + (v, u)]/2$. We shall use the symbol $\perp$ to denote orthogonality in $K_r$.

If $u$, $v$ are in $K$, then by virtue of the positivity of $H$,

$$(U(t)u, v) = \int_m^\infty e^{i\lambda t} d(E_\lambda u, v),$$

with $m$ a fixed positive number. This representation makes it evident that $f(t) = (U(t)u, v)$ can be extended to a holomorphic function in the half-plane $[t + is \mid s > 0]$. Furthermore, $|f(t+is)| \leqq e^{-ms} \|u\| \cdot \|v\|$.

Suppose now that $u$ belongs to $M$, and $v$ belongs to $M^\perp$. By hypothesis $U(t)u$ also belongs to $M$, if $t \geqq t_0$, hence $\operatorname{Re} f(t) = 0$, $t \geqq t_0$. To complete the proof of Theorem 2.1 it suffices to show that such an $f$ must be identically zero. For, if $f \equiv 0$, then $v \perp u$ implies $v \perp U(t)u$ for all $t$. $M$ is a closed subspace of $K_r$, so that for any $t$,

$$U(t)M \subset (M^\perp)^\perp = M. \qquad \text{Q.E.D.}$$

It remains to prove the

LEMMA. *Let $f$ be analytic for* $\operatorname{Im} z > 0$, *continuous for* $\operatorname{Im} z \geqq 0$. *Suppose further that $|f(x+iy)| \leqq Ce^{-my}$ for some $m > 0$ and all $x$. Then* $\operatorname{Re} f(x) = 0$ *for all $x > t_0$ implies that $f = 0$.*

PROOF. Under the hypotheses we may apply the Schwartz reflection principle to continue $f$ analytically as a bounded function in $\operatorname{Re} z > t_0$ by defining $f(x - iy) = -f(x+iy)^*$. It follows that $f(t_0 + iy) = O(e^{-m|y|})$. By a Phragmen-Lindelöf theorem [8, §5.8], $f(z) = 0$ identically in the half-plane, $\operatorname{Re} z \geqq t_0$, hence also in the upper half-plane.

3. **Klein-Gordon equation and domains of uniqueness.** An important physical system in relativistic quantum mechanics is the quantized scalar meson field [6], whose state space of inputs and outputs (the asymptotic free field) is built up from normalizable solutions of the Klein-Gordon equation

$$\square\phi = m^2\phi,$$

where $m>0$ and $\square$ is the wave operator $\Delta-\partial^2/\partial t^2$. The Hilbert space $K_m$ of real normalizable solutions of this equation may be described most succinctly via Fourier analysis as follows (see [5]): Identifying $x_0$ with $t$, we let $x=(x_0, x_1, \cdots, x_n)$ denote a vector in $R^{n+1}$ and $k=(k_0, k_1, \cdots, k_n)$ a vector in the dual space, with $k\cdot x = k_0x_0-k_1x_1- \cdots -k_nx_n$. Consider the complex-valued functions on the hyperboloid $k\cdot k=m^2$ which are measurable and square-summable with respect to the Lorentz-invariant measure $d\chi(k)=|k_0|^{-1}d_nk$, $d_nk$ denoting $n$-dimensional Lebesgue measure $dk_1dk_2\cdots dk_n$. To such a function $f$ corresponds a (generalized) solution $\phi$ given by

$$\phi(x) = \int e^{ik\cdot x}f(k)d\chi(k). \tag{3.1}$$

So that $\phi$ be real-valued, we require that $f(-k)=f(k)^*$; $K_m$ is then the real Hilbert space of all such Hermitian-symmetric, square-summable $f$, with inner product

$$[f, g] = \int f(k)g(-k)d\chi(k),$$

which is always real.

The invariance of the wave operator $\square$ under the inhomogeneous Lorentz group yields an orthogonal representation of this group on $K_m$; in particular, time translations, $x\to x+te_0$, $e_0$ a unit vector on the $x_0$-axis, give rise to a one-parameter orthogonal group

$$U(t): f(k) \to e^{ik_0t}f(k), \qquad f\in K_m.$$

A distinguished property of this representation, for $m>0$, is that a complex structure may be put on $K_m$ such that the representation of the Lorentz group becomes unitary, and $U(t)=e^{itH}$ with self-adjoint generator $H\geqq m$. (See Segal [6].)

Explicitly, let $j$ be the orthogonal transformation on $K_m$ sending $f(k)\to i\,\mathrm{sgn}(k_0)f(k)$; then $j^2=-I$, and $j$ commutes with $U(t)$. By defining a complex inner product

$$(f, g) = [f, g] - i[jf, g]$$

and multiplication by complex scalars

$$(\alpha+i\beta)f = \alpha f + \beta jf,$$

we make $K_m$ into a complex Hilbert space. The operator $H$ is given by multiplication by $|k_0|$, and since $k\cdot k=m^2$, it follows that the spectrum of $H=$range $|k_0|=[m, \infty)$.

THEOREM 3.1. *If $\phi$ is a real-valued normalizable solution of $\Box\phi=m^2\phi$, vanishing on an open time-like cone, then $\phi=0$ identically.*

PROOF. By a time-like cone is meant a cone in $R^{n+1}$ containing a time-like vector $x$ $(x_0^2>x_1^2+\cdots+x_n^2)$. Since $K_m$ is invariant under a change of coordinates in $R^{n+1}$ given by a Lorentz transformation, it is enough to consider the case $\phi=0$ on an open cone $C$ containing the negative $x_0$-axis. If $\phi$ corresponds to $f$ via (3.1), then the hypothesis is equivalent to $[f, P\Psi]=0$ for every real-valued function $\Psi\in C_0^\infty(R^{n+1})$ with compact support in $C$. $P\Psi$ here denotes the projection of $\Psi$ onto $K_m$, i.e. the restriction of the Fourier transform of $\Psi$ (in $R^{n+1}$) to the hyperboloid $k^2=m^2$.

Let now $M=[P\Psi|\Psi\in C_0^\infty(R^{n+1}), \text{Supp }(\Psi)\subset C]^-$, the bar signifying closure in $K_m$. By continuity $[f, g]=0$ for all $g\in M$; furthermore, $M$ is invariant under $[U(t)|\ t\geqq 0]$, since $U(t)P\Psi=P\Psi_t$, with $\Psi_t(x)=\Psi(x+te_0)$. Thus $M$ is a one-sided invariant manifold and by Theorem 2.1 we conclude that $U(t)M\subset M$ for all $t$. For any $\Psi\in C_0^\infty(R^{n+1})$, however, there exists a $t>0$ such that Supp $(\Psi_t)\subset C$. Hence $P\Psi=U(-t)P\Psi_t\in M$, and so $[f, P\Psi]=0$ for all test functions $\Psi$, implying that $f=0$.

It follows from the proof that we have

COROLLARY 3.1. *If $M$ is the set of all $C^\infty$ functions on space-time with compact support contained in a fixed open time-like cone, then the projection of $M$ onto $K_m$ is dense in $K_m$.*

REMARKS. Theorem 3.1 is a sharp result in two directions. In the $m=0$ case, where the energy operator is not strictly positive, there exist nonzero normalizable solutions to the wave equation,

$$\Box\phi = 0,$$

which vanish in the backward light cone. Furthermore, nonzero $C^\infty$ solutions of the Klein-Gordon equation exist which vanish in the backward cone by virtue of familiar general principles concerning hyperbolic equations. (See Courant-Hilbert [1, pp. 450–459] for a discussion of this characteristic Cauchy problem.) Thus we see that the physically-motivated requirements of normalizability and positivity of the energy force a solution to be determined everywhere by its values on any open time-like cone.

**4. Klein-Gordon equation with perturbations.** Our preceding result on domains of uniqueness for normalizable solutions of the KG

(Klein-Gordon) equation is also valid for a class of linear time-independent perturbations, and nonlinear time-dependent perturbations of the equation. In the first case, where the perturbation consists of a non-negative potential $V(x)$, an abstraction of the proof used for the KG equation establishes the result. In the second case, where the perturbation is, e.g., a continuous time-dependent operator small at $t=\pm\infty$, results of Walter Strauss [7] on nonlinear scattering allow the result for the KG equation to be used, but with the weaker conclusion that only the full backward light cone is a domain of uniqueness for normalizable solutions of the nonlinear equation.

Let us consider first the KG equation with potential, viz.,

$$\Box\phi = (m^2 + V)\phi, \tag{4.1}$$

where $V$ is an a.e. non-negative measurable function of the space variable $x$, and $m^2>0$. To avoid irrelevant complications, we shall assume that any singularities of $V$ are mild enough so that the operator $A_0=-\Delta+m^2+V$, with $D(A_0)=\mathcal{S}$ (the Schwartz space of rapidly decreasing functions) is essentially self-adjoint on $K$, the real Hilbert space of real-valued Lebesgue square-summable functions on $R^n$ (see Kato [2]). We denote the closure of $A_0$ by $A$; $A$ is then self-adjoint and is easily seen to satisfy $A\geqq m^2I$. By the spectral theorem $B=A^{1/2}$ exists as a positive self-adjoint operator.

To obtain a Hilbert space structure on solutions of (4.1), we introduce the real Hilbert space $\mathcal{H}_0=D(A)\oplus D(B)$, with the inner product

$$(u, v)_{\mathcal{H}_0} = (Au_1, Av_1)_K + (Bu_2, Bv_2)_K$$

when

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Writing (4.1) in the abstract form

$$d^2\phi/dt^2 = -A\phi, \tag{4.2}$$

we define the space $\mathcal{H}$ of *normalizable solutions* of (4.2) as the set of all $K$-valued functions of $t$ satisfying (i) $t\to\phi(t)$ is strongly differentiable and the derivative $\phi'(t)$ is absolutely continuous and a.e. strongly differentiable, (ii) $\phi(0)\in D(A)$ and $\phi'(0)\in D(B)$, (iii) $\phi$ satisfies (4.2) a.e.

LEMMA 4.1. *Every element of $\mathcal{H}$ has the unique representation*

$$(4.3) \qquad \begin{pmatrix} \phi(t) \\ \phi'(t) \end{pmatrix} = \begin{pmatrix} \cos tB & B^{-1}\sin tB \\ -B\sin tB & \cos tB \end{pmatrix} \begin{pmatrix} \phi(0) \\ \phi'(0) \end{pmatrix}.$$

(See Strauss [7].)

From Lemma 4.1 we see that each element $\phi$ of $\mathcal{H}$ with Cauchy data $\phi(0)=\phi_0$, $\phi'(0)=\phi_1$ at time $t=0$ corresponds uniquely to the element

$$\begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix}$$

of $\mathcal{H}_0$. If the matrix operator in (4.3) is denoted by $U_t$, then $t\rightarrow U_t$ is a continuous one-parameter orthogonal group on $\mathcal{H}_0$ that maps the Cauchy data of $\phi$ at $t=0$ into the Cauchy data of $\phi$ at time $t$.

THEOREM 4.1. *If $\phi$ is a normalizable solution of* (4.2) *and $\mathcal{C}$ is an open cone in $R^{n+1}$ with vertex on the t-axis and containing a semi-infinite segment of the t-axis, then $\phi=0$ on $\mathcal{C}$ implies $\phi\equiv 0$.*

PROOF. By hypothesis

$$(4.4) \qquad \langle \phi, u\rangle \equiv \int_{R^{n+1}} \phi(x,t)u(x,t)\,dxdt = 0$$

for all $u\in C_c^\infty(\mathcal{C})$, and the conclusion of the theorem follows if we show that (4.4) must then hold for all $u\in C_c^\infty(R^{n+1})$. For this purpose we need the following construction:

Let $J$ be the operator on $\mathcal{H}$ which acts formally as the Hilbert transform with respect to time, i.e.

$$J(\cos tB\phi_0 + B^{-1}\sin tB\phi_1) = -\sin tB\phi_0 + B^{-1}\cos tB\phi_1.$$

Equivalently, in terms of its action on the space $\mathcal{H}_0$ of Cauchy data at $t=0$, $J$ corresponds to the matrix operator

$$j = \begin{pmatrix} 0 & B^{-1} \\ -B & 0 \end{pmatrix}.$$

LEMMA 4.2. (a) *$j$ is an isometric transformation on $\mathcal{H}_0$ satisfying $j^*=-j$, $j^2=-I$.*

(b) *$jU_t=U_tj$.*

PROOF. Direct calculation.

By Lemma 4.2, just as in the case of the KG equation in §3, we can define a complex structure on $\mathcal{H}_0$ via $j$, and $t\rightarrow U_t$ becomes a one-parameter unitary group on the complex Hilbert space $\mathcal{H}_c$. The self-adjoint generator $H$ of $U_t$ is obtained as

$$\lim_{t\to 0}(jt)^{-1}(U_t - I) = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix},$$

with $D(H)=D(B^3)\oplus D(B^2)$. Since $H$ commutes with $j$, and $B\geqq mI$ on $K$, it follows that $H\geqq mI$ on $\mathcal{H}_c$.

We now return to the proof of Theorem 4.1. From Theorem 2.1 and the construction above, it follows that the real-linear span in $\mathcal{H}_0$ of the Cauchy data of $\phi$ at times $t$ belonging to any semi-infinite interval coincides with the span of the Cauchy data of $\phi$ at all times $-\infty<t<+\infty$. Since $\phi\to\langle\phi, u\rangle$ (defined in Equation (4.4)) is a continuous linear functional on $\mathcal{H}_0$, we conclude, as in the proof of Theorem 3.1, that $\langle\phi, u\rangle=0$ for all $u\in C_0^\infty(\mathcal{C}_t)$, $-\infty<t<\infty$, where $\mathcal{C}_t$ is the translated cone $te_0+\mathcal{C}$. Any compact set in $R^{n+1}$ is contained in some $\mathcal{C}_t$, however, so $\phi\equiv 0$.

*Nonlinear perturbations.* The case of the equation

$$(\square - m^2)u = L(u) \tag{4.5}$$

with $L$ a possibly nonlinear and time-dependent operator can be successfully treated whenever scattering theory exists for (4.5), considered as a perturbation of the free KG equation

$$(\square - m^2)u = 0.$$

(Strauss [7] has discussed the nonlinear scattering problem, and has obtained some sufficient conditions on $L$ for existence of the wave operators.) Recall that the wave operators $W_\pm$ are constructed as follows: If $u(t)$ is a normalizable solution of (4.5), consider the function $u_s$, obtained as the solution to the KG equation with Cauchy data $u(s)$, $u'(s)$ at time $t=s$. Then $W_\pm u=\lim_{s\to\pm\infty} u_s$, assuming that this strong limit in the Hilbert space of normalizable KG solutions exists.

THEOREM 4.2. *If $W_-$ exists and is* 1-1, *and $u$ is a normalizable solution of* (4.5) *vanishing on the solid backward light cone, then $u=0$.*

*If $W_+$ exists and is* 1-1, *and $u$ is a normalizable solution of* (4.5) *vanishing on the solid forward light cone, then $u=0$.*

This theorem is an immediate consequence of our earlier results and the following

LEMMA. *If $u=0$ on the backward* (*forward*) *light cone, then so does $W_-u$* (*$W_+u$*).

PROOF. Let $\mathcal{C}$ denote the solid backward light cone, $\mathcal{C}_s$ the cone translated through time $s$, and $D_s=\mathcal{C}\cap -\mathcal{C}_s$. With $u_s$ defined as above, the vanishing of $u$ on $\mathcal{C}$ and the hyperbolic propagation property of the KG equation imply that $u_s=0$ on $D_s$ for all $s<0$. Hence $W_{-}u$ vanishes on $\mathcal{C}=\bigcup_{s<0} D_s$.

(The same proof works for $W_{+}u$, of course.)

5. **Causal algebras of field operators.** Consider the quantized free scalar meson field of mass $m>0$. (See [5].) Mathematically, we have a map $f\rightarrow R(f)$ from $C_c^\infty(R^4)$ to self-adjoint operators on a complex Hilbert space $K$ satisfying the usual physical desiderata (commutation rules, Lorentz-transformation properties, irreducibility) and certain continuity requirements. Let $W(f)=\exp[iR(f)]$, and denote by $Pf$ the projection of $f\in C_c^\infty(R^4)$ onto the KG Hilbert space, i.e. the restriction of the Fourier transform of $f$ to the mass hyperboloid $k^2=m^2$. By hypothesis, the set of operators $[W(f)]$ is irreducible on $K$, and the map $f\rightarrow W(f)$ is continuous with respect to the Lorentz-invariant Hilbert topology on $Pf$ and the weak operator topology on $W(f)$.

THEOREM 5.1. *If $R(\cdot)$ is the quantized field for the* KG *equation of mass $m>0$, then the operators $R(g)$, with* Supp $(g)$ *contained in a fixed open time-like cone, generate all bounded operators on the field state space $K$.*

PROOF. From Corollary 3.1, the set of all such $g$ is strongly dense in the KG Hilbert space $K_m$. Hence by the continuity of the map $f\rightarrow W(f)$, it follows that the ring of operators generated by $W(g)$, $g$ ranging over a dense subset of $K_m$, is $B(K)$.

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