ON LOCALLY ISOMETRIC MAPPINGS OF A G-SPACE ON ITSELF

W. A. KIRK

In his study [2] of locally isometric mappings of a G-space $\bar{R}$ on a G-space R, Busemann considers the following question: Under what conditions is every locally isometric mapping of a G-space R on itself a motion? He proves that such is the case if either (i) the fundamental group of R is not isomorphic to a proper subgroup of itself, or (ii) R is compact. Busemann suggests [2, p. 405] that conditions (other than (i)) be sought which apply to noncompact spaces, in particular, conditions which apply to an ordinary cylinder. Szenthe replies to this in a recent paper [3] in which he finds conditions in terms of certain bounds on the lengths of nonoverlapping geodesic curves which begin and end at the same point. In [1] Busemann shows that under appropriate hypotheses on the order of magnitude of volumes of spheres a locally isometric mapping of a noncompact G-space on itself is a motion. Our paper provides another condition. We first show that if a locally isometric mapping of R on itself has a fixed point, then it is a motion. From this it readily follows that if the motions of R form a transitive group, then *every* locally isometric mapping of R on itself is a motion.

Let ϕ denote a locally isometric mapping of a G-space $\bar{R}$ on a G-space R. The terminology we use and the following properties of ϕ are found in Busemann [2, §27].

(1) *If $\bar{x}(\tau)$, $\alpha \leqq \tau \leqq \beta$, is a curve in $\bar{R}$ and if $\phi(\bar{x}(\tau)) = x(\tau)$ represents a segment, then $\bar{x}(\tau)$ represents a segment and $\bar{x}(\alpha)\bar{x}(\beta) = x(\alpha)x(\beta)$.*

(2) *For a given curve $x(\tau)$, $\alpha \leqq \tau \leqq \beta$, in R and a given point $\bar{a}$ of $\bar{R}$ such that $\phi(\bar{a}) = x(\alpha)$ there is exactly one curve $\bar{x}(\tau)$ in $\bar{R}$ such that $\phi(\bar{x}(\tau)) = x(\tau)$ with $\bar{x}(\alpha) = \bar{a}$.*

(3) *There is a number $\rho(p) > 0$ such that if $\phi(\bar{p}_1) = \phi(\bar{p}_2) = p$, $\bar{p}_1 \neq \bar{p}_2$, then $\bar{p}_1\bar{p}_2 \geqq 2\rho(p)$.*

(4) *The number of points of $\bar{R}$ which lie over a given point of R is countable and is the same for different points of R.*

(5) *If ϕ is* 1-1 *then ϕ is an isometry.*

Since each two points of a G-space are joined by a metric segment of the space, the following is an immediate consequence of (1) and (2).

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(6) *If a and b are any two points of R and if $\bar{a}$ is any point of $\overline{R}$ which lies over a, then there is a point $\bar{b}$ of $\overline{R}$ which lies over b, and $\bar{a}\bar{b} = ab$.*

It follows from the definition that

(7) *If ϕ is a locally isometric mapping of R on itself then ϕ^n is also* $(n = 1, 2, \cdots)$.

THEOREM 1. *If ϕ is a locally isometric mapping of a G-space R on itself and if ϕ has a fixed point, then ϕ is a motion.*

PROOF. Suppose the contrary, and let p denote any fixed point of ϕ. By (5) ϕ is not 1-1 so by (4) there is a point $p_1 \neq p$ such that $\phi(p_1) = p$. By (6) there is a point p_2 such that $\phi(p_2) = p_1$ and $pp_2 = pp_1$. We define inductively a sequence $\{p_n\}$ of points of R such that $\phi(p_{n+1}) = p_n$ and $pp_n = pp_1$, $n = 1, 2, \cdots$.

If $n < m$ then $\phi^n(p_n) = p$ while $\phi^n(p_m) = p_{m-n}$. Since $pp_{m-n} = pp_1$, we see that $p_{m-n} \neq p$. Therefore $\phi^n(p_n) \neq \phi^n(p_m)$ and $p_n \neq p_m$. This shows that the elements of $\{p_n\}$ are pairwise distinct. By (7) ϕ^n, for each positive integer n, is a locally isometric mapping of R on itself, so by (3) $p_i p_j \geqq 2\rho(p)$ if $i \neq j$. This contradicts the finite compactness of R.

THEOREM 2. *A locally isometric mapping ϕ of a G-space R on itself is a motion if and only if there is a motion ψ of R such that for some point $p \in R$, $\psi(\phi(p)) = p$.*

PROOF. The necessity is trivial. The sufficiency is established by observing that $\psi\phi$ is a locally isometric mapping of R on itself with fixed point p. By Theorem 1, $\psi\phi$ is a motion and hence 1-1. Therefore, ϕ is 1-1 and by (5) a motion of R.

The motions of a G-space form a transitive group if, given any two points of the space, there is a motion of the space which maps one into the other. Thus the following theorem is a corollary to Theorem 2.

THEOREM 3. *If a G-space R has a transitive group of motions, then every locally isometric mapping of R on itself is a motion.*

Apparently little is known about the problem of determining the G-spaces with transitive groups of motion in general, but the 2-dimensional case has been completely solved. Busemann has shown [**2**, p. 371] that the only G-surfaces (2-dimensional G-spaces) with transitive groups of motions are: The plane with a Minkowskian or quasi-hyperbolic metric, the cylinder and torus with a Minkowskian metric, the sphere and projective plane with a spherical metric.

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UNIVERSITY OF CALIFORNIA, RIVERSIDE

A-GENUS AND INDECOMPOSABILITY OF DIFFERENTIABLE MANIFOLDS

YASURÔ TOMONAGA

Introduction. In the previous paper [1] we have studied the conditions on which a differentiable manifold be indecomposable and cited many examples of indecomposable manifolds. In this paper we shall study the relations between A-genus and indecomposability of a differentiable manifold.

1. Hereafter we denote by X_n an n-dimensional compact orientable differentiable manifold. If $X_n = X_r \cdot X_s$ we say that X_n is *decomposable* and if not, we say that X_n is *indecomposable*. If $X_{4n} = X_n \cdot X_s$ we have

$$A(X_{4n}) = A(X_r)A(X_s); \tag{1.1}$$

where $A(X)$ denotes the A-genus of X and we define as follows:

$$A(X_n) = 0, \qquad n \not\equiv 0 \bmod 4. \tag{1.2}$$

If r and s are divisible by 4, the relation (1.1) follows from the general property of multiplicative series [2, p. 75]. According to the cobordism theory, the cobordism components of X_r ($r \not\equiv 0 \bmod 4$) consist only of torsions. Hence the product $X_r \cdot X_s$ also consists only of torsions. Therefore $A(X_r \cdot X_s)$ is zero. Thus (1.1) holds in general. Meanwhile Atiyah and Hirzebruch have proved the following:

THEOREM 1 (ATIYAH AND HIRZEBRUCH [3]). *If X_{4n} is differentiably imbedded in the $(8n-2q)$-sphere, then $A(X_{4n})$ is divisible by 2^{q+1}. If moreover $q \equiv 2 \bmod 4$, then $A(X_{4n})$ is divisible by 2^{q+2}.*

It is well known that an X_n is always differentiably imbedded in the $2n$-sphere. Hence we have from the above theorem

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