Successive integrations by parts and application of the conditions $f^m(1)=0$ gives $I(x)=f(x)$.

The second solution can be verified in a similar way.

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INFINITE ORDER DIFFERENTIAL EQUATIONS

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1. **Introduction.** Let $f(z)=\sum_{k=0}^{\infty} A_k z^k$ converge for $|z|<R$, where $0<R\leqq\infty$; let E be the vector space of entire functions of exponential type less than R; and let $\mathfrak{D}=\sum_{k=0}^{\infty} A_k D^k$, where D is the differential operator. The purpose of this paper is to provide a brief derivation of the results of Muggli [2, p. 154] regarding the general solutions in E of the equations

$$\mathfrak{D}\phi = 0, \quad \text{and} \tag{1}$$

$$\mathfrak{D}\phi = \psi. \tag{2}$$

It will be shown that $\mathfrak{D}$ is a surjective endomorphism of E, reducing the problem of solving (2) to that of solving (1). It is easy to show that if ζ is a zero of f of order at least $h+1$ and of modulus less that R, then $z^h e^{\zeta z}$ is a solution of (1). If B is the set of all such exponential monomials, then Muggli's result says that B is a basis for the solutions of (1) and that each solution ϕ is representable as a sum of exponential monomials with exponent coefficients in the conjugate indicator diagram of ϕ. Each solution of (2) is then representable as the sum of a contour integral and a solution of (1).

Results similar to these have been obtained by Sheffer [3, p. 255]

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and Sikkema [4, p. 203]. In case f itself is entire and of exponential type, the solutions of (1) are entire mean periodic functions and have been studied widely. The result concerning (1) may be viewed as a generalization of the fact that any function of exponential type σ and of period 2π can be expressed in the form $\sum c_n e^{inz}$, where the sum is over integers n with $|n| \leqq \sigma$. The explicitness of the result given here and its brief derivation are due to the use of a lemma which is perhaps of interest in itself.

2. **Preliminaries.** Let C denote the disk $|z| < R$, and let P' denote the complement in the plane of the set P. For ϕ in E, $P(\phi)$ will denote the conjugate indicator diagram of ϕ, and $L\phi$ will denote the Borel transform [1, p. 73] of ϕ. Then $P(\phi) \subset C$, and $L\phi$ is analytic in $P'(\phi)$. Also if $t \in C$, then $\mathfrak{D}_z e^{zt} = f(t)e^{zt}$, where $\mathfrak{D}_z = \sum_{k=0}^{\infty} A_k \partial^k/\partial z^k$.

The fact that $\mathfrak{D}$ maps E into E with $P(\mathfrak{D}\phi) \subset P(\phi)$ follows easily from the Pólya representation of ϕ [1, p. 74]. For if γ is a simple closed (rectifiable and positively oriented) curve in C containing $P(\phi)$ in its interior, then

$$\mathfrak{D}\phi(z) = (2\pi i)^{-1} \int_\gamma e^{zw} f(w) L\phi(w) dw,$$

the change in order of integration and summation being justified by the uniform convergence of this integrand on γ. $\mathfrak{D}\phi$ is obviously entire, and since γ may be chosen to be arbitrarily close to the boundary of $P(\phi)$, it easily follows that the indicator function of $\mathfrak{D}\phi$ is less than or equal to that of ϕ and so $P(\mathfrak{D}\phi) \subset P(\phi)$.

$\mathfrak{D}$ also maps E onto E; in fact if $\psi \in E$, then there is a ψ_0 in E such that $\mathfrak{D}\psi_0 = \psi$ and $P(\psi_0) = P(\psi)$. For suppose that γ is a simple closed curve in C about $P(\psi)$ such that there are no zeros of f on γ or in the region common to the interior of γ and $P'(\psi)$. Let

$$\psi_0(z) = \frac{1}{2\pi i} \int_\gamma \frac{e^{zw} L\psi(w)}{f(w)} dw. \tag{3}$$

Then $\psi_0 \in E$ and since ψ_0 is independent of the choice of γ for γ in the zero free region mentioned, $P(\psi_0) \subset P(\psi)$. But easily $\mathfrak{D}\psi_0 = \psi$, and since $P(\psi) = P(\mathfrak{D}\psi_0) \subset P(\psi_0)$, $P(\psi_0) = P(\psi)$.

3. **Representation of solutions.**

Definition. If $\phi \in E$ and $t \in C$, then

$$T\phi(t) = \left[\mathfrak{D}_z \int_0^z \phi(s) e^{(z-s)t} ds \right]_{z=0}.$$

$T\phi$ is well defined; for if $t \in C$, then $e^{zt} \in E$ and its convolution with ϕ has as a type the maximum of $|t|$ and the type of ϕ. Hence the convolution and its image under $\mathfrak{D}_z$ are in E. Further, $T\phi$ is analytic in C. For suppose N is a closed circular neighborhood in C and $t \in N$. Then the convolution of ϕ and e^{zt} is of a type $\sigma < R$, and its Borel transform is $(z-t)^{-1}L\phi(z)$. By choosing a circle β about the origin with radius between σ and R and which contains N in its interior, we may write the convolution as a Pólya integral over β and obtain

$$(4) \qquad T\phi(t) = \int_\beta H(w, t)dw, \quad \text{where} \quad H(w, t) = \frac{f(w)L\phi(w)}{2\pi i(w-t)} \cdot$$

Application of Morera's theorem yields the analyticity of $T\phi$ in the interior of N.

LEMMA. *If $\phi \in E$ and $t \in C \cap P'(\phi)$, then $T\phi(t) = f(t)L\phi(t) - L\mathfrak{D}\phi(t)$.*

PROOF. Let ϕ be of type $\sigma < R$ and $\epsilon = (R-\sigma)/3$. Let α, β, and γ be the circles $|z| = \sigma + \epsilon$, $|z| = \sigma + 2\epsilon$, and $|z - (\sigma + R)/2| = \epsilon/3$, respectively. The left member of the identity to be established is analytic in C, while the right member is analytic in $C \cap P'(\phi)$ since $P(\mathfrak{D}\phi) \subset P(\phi)$. Since both are uniform, it suffices to establish the identity for t inside γ. Considering only such t, write $\phi(t)$ as a Pólya integral over α, and then operate on this with $\mathfrak{D}$ and then with L, writing the latter as a Laplace transform from 0 to ∞. Then with $H(w, t)$ as in (4), $L\mathfrak{D}\phi(t) = -\int_\alpha H(w, t)dw$, the change in order of integration being justified by the observation that $\int_0^\infty \exp[(w-t)s]ds$ converges absolutely since $\Re(t-w) > 0$ for w on α. Write $\int_\alpha H(w, t)dw$ as the difference of integrals over β and γ. As in (4), the integral over β is $T\phi(t)$ while the integral over γ is $f(t)L\phi(t)$.

DEFINITION. Let $\{\zeta_k\}_{k \in K}$ be the zeros of f in C with m_{k+1} the order of ζ_k. Let c_k be a circle in C about ζ_k containing no other zeros in or on itself. For each $k \in K$ and natural h, define the linear functional $\mathfrak{D}_{kh}$ on E by

$$\mathfrak{D}_{kh}\phi = \frac{1}{2\pi i}\int_{c_k} \frac{(t-\zeta_k)^h T\phi(t)}{f(t)} dt.$$

Using the lemma, it is easy to show that $\mathfrak{D}_{kh}(z^p \exp \zeta_q z) = h!\delta_{ph}\delta_{qk}$ when ζ_q is a zero of f of order at most $p+1$ and hence the elements in B are linearly independent.

THEOREM. *Let $\phi \in E$; then $\mathfrak{D}\phi = 0$ if and only if*

$$(5) \qquad \phi(z) = \sum_k e^{\zeta_k z} \sum_{h=0}^{m_k} \frac{\mathfrak{D}_{kh}\phi}{h!} z^h,$$

where k ranges over those k for which $\zeta_k \in P(\phi)$.

Proof. The fact that such a sum satisfies the equation is obvious. Suppose that $\mathfrak{D}\phi = 0$. Upon writing the sum in (5) with the coefficients in integral form, the m_k may be replaced by infinity since the terms so introduced are zero by Cauchy's theorem, and then the sum over h may be replaced by an exponential function. The sum of the integrals over the c_k may be replaced by one integral over a closed path γ in C about $P(\phi)$ whose interior contains no zeros of f other than those in $P(\phi)$. Using the lemma and the fact that $\mathfrak{D}\phi = 0$, the resulting integral is the Pólya representation of ϕ.

Corollary. *Let ϕ and $\psi \in E$; then $\mathfrak{D}\phi = \psi$ if and only if*

$$\phi(z) = \psi_0(z) + \sum_k e^{\zeta_k z} \sum_{h=0}^{m_k} \frac{\mathfrak{D}_{kh}(\phi - \psi_0)}{h!} z^h,$$

where k ranges over those k for which $\zeta_k \in P(\phi - \psi_0)$ and ψ_0 is given by (3).

Other interesting results follow from the theorem and its corollary by observing their implications when $P(\phi)$ or $P(\phi - \psi_0)$ contains one or no zeros of f, as is the case when one of the diagrams is a point. It follows from the theorem that any solution of (1) in E is actually the solution of a similar equation of finite order having a characteristic function dividing f in the ring of functions analytic in C.

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