REMARKS ON THE CLASSIFICATION OF RIEMANN SURFACES[1]

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1. **Introduction.** In [5] Royden constructed two Riemann surfaces to show that neither of the classes O_{AB} and O_{AD} of Riemann surfaces is quasiconformally invariant. In the present note it is shown how a slight modification of Royden's argument serves to establish the more general

THEOREM 1. *No class of Riemann surfaces that contains O_L and is contained in O_{AD} is quasiconformally invariant.*

This result is then used to prove

THEOREM 2. *There is no inclusion relation between either of O_L and $O_{L'}$ and any one of O_{HD}, O_{FD} and O_{FB}.*

2. **Notation and background.** For any class T of functions that can be considered on a Riemann surface, let O_T stand for the class of all Riemann surfaces that do not admit nonconstant members of T. Denote by L the class of Lindelöfian meromorphic functions (see [2]), i.e. meromorphic functions of bounded characteristic, and by L' the class of those members of L which are pole-free. AB and AD denoting, respectively, the classes of bounded and Dirichlet-bounded analytic functions, it is known (see [2, p. 442] and [1, p. 201 and p. 256]) that

$$O_L \subsetneqq O_{L'} \subsetneqq O_{AB} \subsetneqq O_{AD}. \tag{1}$$

Let, now, HD denote the class of Dirichlet-bounded harmonic functions and FB and FD signify, respectively, the classes of bounded and Dirichlet-bounded harmonic functions whose conjugate periods vanish along dividing cycles. It is clear that

$$O_{HD} \subset O_{FD} \subset O_{AD}, \tag{2}$$

and it is known [7, p. 469] that

$$O_{FB} \subset O_{FD}. \tag{3}$$

3. **The examples.** Denote by S the region of the complex plane obtained by punching out $z=0$ and $z=2$ from the disc $|z|<3$. Let

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W_1 be a two-sheeted conformal cover of S having branch points over $z=1/n$ and $z=2-1/n$, $n=2, 3, \cdots, \infty$, and such that two curves of W_1 lie over the unit circle in S. Evidently, the projection map from W_1 to S is an AD function, i.e.

$$W_1 \notin O_{AD}. \tag{4}$$

Choose, now, s, an irrational multiple of 2π. Cut the upper sheet of W_1 over the unit circle $|z|=1$ and identify the point over e^{it} on the inside of the cut with the point over e^{it+is} on the outside. Denote the resulting Riemann surface by W_2. We claim that

$$W_2 \in O_L. \tag{5}$$

For, otherwise, let $f \in L$ on W_2. The projection map from the subregion W_2' of W_2 over the annulus $0<|z|<1$ into this annulus is of constant valence two. Hence, by a theorem of Heins [**2**, p. 444], f is the quotient of two bounded analytic functions, say, g and h, on W_2'. But then, the argument of Royden [**5**, p. 6] shows that g and h are single valued functions of z for $|z|<1$, and hence, so is f. Similarly, f is a single valued function of z for $1<|z|<3$. Hence, over $|z|=1$, we have $f(e^{it})=f(e^{it+is})$.

Iterating we obtain that

$$f(e^{it}) = f(e^{it+ims}), \qquad m = 1, 2, \cdots, \infty.$$

Since the set of points $\{e^{it+ims}\}$ is dense in the unit circle, this implies that f is a constant, which is a contradiction that establishes (5).

Now, the map ϕ: $W_1 \to W_2$, obtained by taking ϕ to be the identity except on the annulus over $1/2<|z|<1$ in the upper sheet of W_1 and $\phi(re^{it})=re^{it+is(2r-1)}$ on this annulus, is a quasiconformal homeomorphism of W_1 onto W_2. This observation, together with (4) and (5), establishes Theorem 1.

REMARKS. (i) As noted by Royden [**5**, p. 6], the map ϕ is "ultimately" conformal. Hence, the property of belonging to any of the classes of Theorem 1 is not a property of the ideal boundary [**6**, p. 58].

(ii) Examples of classes of Riemann surfaces for which Theorem 1 is applicable are O_L, $O_{L'}$, O_{AB}, O_{AD} and the classes O_{AM_α} [**3**, p. 179].

(iii)[2] A construction, similar to the one above, carried out over the Riemann sphere instead of the disc $|z|<3$, yields an example to show that the class of parabolic Riemann surfaces admitting meromorphic functions of bounded valence is not preserved under quasiconformal maps.

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[2] The author is indebted to the referee for pointing this out.

4. **Proof of Theorem 2.** It is known [6, p. 57] that O_{FD} is quasiconformally invariant. This, in view of (2) and Theorem 1, implies that

$$O_L \not\subset O_{FD}, \qquad O_L \not\subset O_{HD}. \tag{6}$$

Also, there exists [2, p. 441] a planar surface which does not belong to $O_{L'}$ but belongs to O_{AB}. Since every cycle on a planar surface is dividing, this implies that

$$O_{FB} \not\subset O_{L'}. \tag{7}$$

Finally, in view of Theorem 26H of [1, p. 264] and (1), we have

$$O_{HD} \not\subset O_{L'}. \tag{8}$$

Combining the first part of (1) with (6)–(8), we obtain the result.

NOTE. An alternative proof of the second part of (6) was given in [4].

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