# A COUNTEREXAMPLE TO A PROBLEM OF SZ.-NAGY[1]

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The purpose of this note is to construct an operator, in a Hilbert space, with uniformly bounded powers, which is not similar to a contraction. (An operator $P$ is a contraction if $\|P\| \leqq 1$.) This will furnish a negative answer to a problem raised in [1].

Let $H$ be a Hilbert space and $T$ an operator such that $\|T^n\| \leqq M$ $(n=1, 2, \cdots)$. Define

$$H_0(T) = \{x \mid \text{weak lim } T^n x = 0\}, \qquad H_1(T) = H_0(T^*)^\perp.$$

We proved in [2, Theorem 3.1] that if $P$ is a contraction then $H_0(P) = H_0(P^*)$. Let $T = SPS^{-1}$ where $\|P\| \leqq 1$, then:

$$H_0(T) = \{x \mid P^n S^{-1} x \xrightarrow{\omega} 0\} = \{x \mid S^{-1}x \in H_0(P)\} = S(H_0(P))$$

and

$$\begin{aligned} H_1(T) &= H_0(T^*)^\perp = [S^{*-1}(H_0(P^*))]^\perp = \{x \mid S^{-1}x \perp H_0(P^*)\} \\ &= S(H_1(P)). \end{aligned}$$

But $H_1(P) \perp H_0(P)$ and thus $H_0(T) \cap H_1(T) = 0$.

We will construct an operator, with uniformly bounded iterates, for which $H_0 \cap H_1 \neq 0$, and thus the operator is not similar to a contraction.

Let $H = K \oplus L$, where $K$ is generated by the orthonormal sequence $\{e_i\}$ and $L$ by the orthonormal sequence $\{f_i\}$. Let $\{n_k\}$ be a subsequence of the integers which is "sparse" in the sense that:

$$n_{k+1} - n_k > 2n_k$$

(e.g., $n_k = 4^k$).

NOTATION. *Integers in the sequence $\{n_k\}$ will be denoted by $\alpha, \beta, \cdots$. Integers outside the sequence $\{n_k\}$ will be denoted by $a, b, \cdots$. The letters $i, j, k \cdots$ will stand for integers which might be in or out of the exceptional sequence $\{n_k\}$.*

DEFINITION. *Let the operator $T$ be defined by the linear extension of*

$$Te_1 = 0, \qquad Te_i = e_{i-1}, \qquad i \geqq 2,$$

*and*

Received by the editors December 21, 1962 and, in revised form, May 2, 1963.

[1] The research in this document has been sponsored in part by the Air Force Office of Scientific Research, OAR, through the European Office Aerospace Research, United States Air Force.

$$Tf_a = f_{a+1}, \qquad Tf_\alpha = e_\alpha + f_{\alpha+1}.$$

LEMMA 1. *For each $j$ and $h$*

$$T^j f_h = f_{h+j} + \epsilon(j, h) e_{i(j,h)}$$

*where*:

a. *If no element of the sequence $n_k$ is in $[h, h+j)$ then $\epsilon(j, h) = 0$.*

b. *Let $\alpha$ be the largest element of $\{n_k\}$ with $h \leqq \alpha < h+j$ then*:

$$\epsilon(j, h) = 0 \qquad \text{if } 2\alpha < j + h,$$
$$\epsilon(j, h) = 1 \qquad \text{if } 2\alpha \geqq j + h$$

*and*

$$i(j, h) = 2\alpha - j - h + 1.$$

PROOF. Let us prove by induction on $j$. For $j=1$ the proof is clear. Assume the lemma holds for $j$. Now if no element of $\{n_k\}$ is in $[h, h+j)$ then there are two possibilities:

1. If $h+j \notin \{n_k\}$ then

$$T^{j+1} f_h = T(T^j f_h) = Tf_{h+j} = f_{j+h+1} \quad \text{and} \quad \epsilon(j+1, h) = 0.$$

2. If $h+j = \alpha \in \{n_k\}$ then

$$T^{j+1} f_h = Tf_\alpha = f_{\alpha+1} + e_\alpha = f_{h+j+1} + e_{2\alpha-j-h}.$$

On the other hand, let $\alpha$ be the largest element of $\{n_k\}$ with $h \leqq \alpha < h+j$. Again there are two possibilities:

1. If $h+j = \beta \in \{n_k\}$ then $h+j = \beta > 2\alpha$ and $\epsilon(j, h) = 0$ hence

$$T^{j+1} f_h = Tf_{j+h} = Tf_\beta = f_{j+h+1} + e_{j+h} = f_{j+h+1} + e_{2\beta-j-h}.$$

2. If $h+j \notin \{n_k\}$ then

$$\begin{aligned} T^{j+1} f_h &= T(T^j f_h) = T(f_{j+h} + \epsilon(j, h) e_{i(j,h)}) \\ &= f_{j+h+1} + \epsilon(j, h) T e_{i(j,h)}. \end{aligned}$$

Now if $2\alpha < j+h$ then $\epsilon(j, h) = 0$ and $\epsilon(j+1, h) = 0$ since $\alpha$ is the largest element of $\{n_k\}$ in $[h, j+h+1)$ and $2\alpha < j+h+1$. However if $2\alpha \geqq j+h$ then

$$T^{j+1} f_h = f_{j+h+1} + Te_{2\alpha-j-h+1} = f_{j+h+1} + e_{2\alpha-j-h},$$

where $e_0 = 0$.

Thus $\epsilon(j+1, h) = 1$ if $2\alpha \geqq j+h+1$ and then

$$i(j+1, h) = 2\alpha - j - h.$$

while $\epsilon(j+1, h) = 0$ if $2\alpha = j+h$.

LEMMA 2. *Let $j$ be given and $h_1 \neq h_2$. If $\epsilon(j, h_1) = \epsilon(j, h_2) = 1$ then $i(j, h_1) \neq i(j, h_2)$.*

PROOF. Let $\alpha_1$ and $\alpha_2$ be the largest elements of $\{n_k\}$ in $[h_1, j+h_1)$ and $[h_2, j+h_2)$ respectively. If $\alpha_1 = \alpha_2$ then

$$i(j, h_1) = 2\alpha_1 - j - h_1 + 1 \neq 2\alpha_1 - j - h_2 + 1 = i(j, h_2).$$

If $\alpha_2 > \alpha_1$ then

$$2\alpha_1 > i + h_1 > j,$$

since $\epsilon(j, h_1) = 1$. Thus

$$i(j, h_2) = 2\alpha_2 - j - h_2 + 1 \geqq \alpha_2 - j \geqq \alpha_2 - 2\alpha_1 > \alpha_1$$

but

$$\alpha_1 \geqq 2\alpha_1 - j - h_1 = i(j, h_1),$$

since $\alpha_1 < j + h_1$ by its definition.

LEMMA 3. $\|T^j\| \leqq 2$.

PROOF. Let $y = \sum a_h f_h \in L$; then

$$\|T^j y\|^2 = \|\sum a_h f_{j+h}\|^2 + \|\sum a_h \epsilon(j, h) e_{i(j,h)}\|^2$$

and since the indices $i(j, h)$ are different, for different values of $h$, we get

$$\|T^j y\|^2 \leqq 2 \sum |a_h|^2 = 2\|y\|^2.$$

Let $z \in H$; then $z = x + y$ where $x \in K$ and $y \in L$. Then

$$\begin{aligned}\|T^j z\|^2 &\leqq (\|T^j x\| + \|T^j y\|)^2 \leqq (\|x\| + \sqrt{2}\|y\|)^2 \\ &\leqq 2(\|x\| + \|y\|)^2 \leqq 4(\|x\|^2 + \|y\|^2) = 4\|z\|^2.\end{aligned}$$

LEMMA 4. *The vector $e_1$ belongs to both $H_0(T)$ and $H_1(T)$.*

PROOF. Clearly $e_1 \in H_0(T)$. Now $T^{2n_k - 1} f_1 = f_{2n_k} + e_1 \to^{\omega} e_1$. Thus if $z \in H_0(T^*)$ then

$$(e_1, z) = \lim (T^{2n_k - 1} f_1, z) = \lim (f_1, T^{*2n_k - 1} z) = 0,$$

since weak lim $T^{*n} z = 0$. Hence $e_1 \in H_0(T^*)^\perp = H_1(T)$.

## BIBLIOGRAPHY

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