ON FOGUEL'S ANSWER TO NAGY'S QUESTION

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Nagy's question is whether or not every power-bounded operator is similar to a contraction [3]. ("Power-bounded" means that the norms of the positive powers are bounded.) Foguel's answer is no [1]. The purpose of this note is to look at Foguel's ingenious counterexample from a point of view somewhat different from his own. The advantage of the new look is that it is less computational; its drawback is that the intuitive motivation is less transparent.

Let H_0 be a Hilbert space with an orthonormal basis $\{e_0, e_1, e_2, \cdots\}$, and let S be the unilateral shift on H_0 ($Se_n = e_{n+1}$, $n=0, 1, 2, \cdots$). Let J be an infinite set of natural numbers that is "sparse" in the sense that if i and j belong to J and $i<j$, then $2i<j$. (Example: J can be the set of positive integral powers of 3.) Let Q be the projection from H_0 onto the span of all the e_j's with j in J. If H is the direct sum of two copies of H_0 (the set of all ordered pairs $\langle f, g\rangle$ with f and g in H_0), then every operator on H is given by a two-by-two matrix whose entries are operators on H_0. Principal assertion: if

$$A = \begin{pmatrix} S^* & Q \\ 0 & S \end{pmatrix},$$

then A is power-bounded, but A is not similar to a contraction.

A trivial induction shows that

$$A^n = \begin{pmatrix} S^{*n} & Q_n \\ 0 & S^n \end{pmatrix},$$

where $Q_0 = 0$ and $Q_{n+1} = \sum_{i=0}^{n} S^{*n-i}QS^i$, $n=0, 1, 2, \cdots$. To prove that A is power-bounded is the same as to prove that the norms of the Q's are bounded. It turns out, in fact, that each Q is a partial isometry whose range is spanned by a set of e's. To prove this, consider $Q_{n+1}e_m = \sum_{i=0}^{n} S^{*n-i}Qe_{m+i}$. If $n-i>m+i$, then $S^{*n-i}Qe_{m+i}=0$, because either $m+i \notin J$ (in which case $Qe_{m+i}=0$), or $m+i \in J$ (in which case S^{*n-i} annihilates e_{m+i}). Among the remaining values of i (the ones for which $i \leqq n \leqq m+2i$) at most one can be such that $m+i \in J$. Reason: if both i and j have these properties, and, say, $i<j$, then $m+i<m+j$, so that $2(m+i)<m+j$, or $m+2i<j$, which

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contradicts the relation $j \leqq n \leqq m+2i$. Conclusion: $Q_{n+1}e_m$ is either 0 or e_{m+2i-n}; it is the latter just in case there exists an i (necessarily unique) such that $i \leqq n \leqq m+2i$ and $m+i \in J$. This conclusion will be used again presently; its function so far was to prove that A is power-bounded.

It remains to prove that A is not similar to a contraction. For this purpose Foguel introduces the set $Z(A)$ of all those vectors f in H for which $A^n f \to 0$ weakly as $n \to \infty$. (Here H can be an arbitrary Hilbert space and A an arbitrary operator on it.) The pertinent lemma is that if A is similar to a contraction, then $Z(A) \cap (Z(A^*))^\perp = \{0\}$. (A proof of the lemma appears below.) The conclusion of the preceding paragraph makes it possible to apply the lemma, as follows. If $j \in J$, then $Q_{2j+1}e_0 = e_0$. Since $A^{2j+1}\langle 0, e_0\rangle = \langle Q_{2j+1}e_0, S^{2j+1}e_0\rangle = \langle e_0, e_{2j+1}\rangle$, so that $A^{2j+1}\langle 0, e_0\rangle \to \langle e_0, 0\rangle$ weakly as $j \to \infty$ (through values in J), it follows that if $\langle f, g\rangle \in Z(A^*)$ (that is, if $A^{*n}\langle f, g\rangle \to \langle 0, 0\rangle$ weakly as $n \to \infty$), then

$$(\langle e_0, 0\rangle, \langle f, g\rangle) = \lim_{j \in J} (A^{2j+1}\langle 0, e_0\rangle, \langle f, g\rangle) = \lim_{j \in J} (\langle 0, e_0\rangle, A^{*2j+1}\langle f, g\rangle) = 0,$$

so that $\langle e_0, 0\rangle \in (Z(A^*))^\perp$. Since, however, $A\langle e_0, 0\rangle = \langle 0, 0\rangle$, the vector $\langle e_0, 0\rangle$ belongs to $Z(A)$ also, and consequently A cannot be similar to a contraction.

For the lemma Foguel refers to an earlier paper. Here is an alternative approach, via the theory of strong unitary dilations [2].

(1) If U is unitary, then $Z(U) \subset Z(U^*)$. Indeed, represent U as multiplication by a measurable function ϕ of constant modulus 1 on some $L^2(\mu)$. It is to be proved that if $\int \phi^n f \bar{g} d\mu \to 0$ for every g, then $\int \bar{\phi}^n f \bar{h} d\mu \to 0$ for every h. To prove it, given h, put $g = (\operatorname{sgn} f)^2 \bar{h}$, and form the complex conjugate of the hypothesis.

(2) If C is a contraction, then $Z(C) \subset Z(C^*)$. To prove this, let U be a minimal strong unitary dilation of C. That is: if C operates on H, then U operates on a larger Hilbert space K; if P is the projection from K onto H, then $C^n f = PU^n f$ for all f in H ($n = 1, 2, 3, \cdots$). For each f in $Z(C)$, let K_f be the set of all those g in K for which $(U^n f, g) \to 0$. Since $f \in Z(C)$, it follows that $H \subset K_f$; indeed, if $g \in H$, then $(U^n f, g) = (C^n f, g)$. It is trivial that K_f is a linear manifold; the power-boundedness of U implies that K_f is closed. Since K_f is invariant under both U and U^*, the minimality of U implies that $K_f = K$ for each f in $Z(C)$. This implies that $Z(C) \subset Z(U)$, and hence, by (1), that $Z(C) \subset Z(U^*)$. Since U^* is a strong dilation of C^*, it follows that $Z(C) \subset Z(C^*)$.

The promised lemma is now within reach. If A is similar to a contraction C, say $A = TCT^{-1}$, then it is easy to verify that $Z(A) = TZ(C)$ and $(Z(A^*))^\perp = T(Z(C^*))^\perp$. Since, by (2), $Z(C) \cap (Z(C^*))^\perp = \{0\}$, the conclusion $Z(A) \cap (Z(A^*))^\perp = \{0\}$ follows by an application of T.

References

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