ON NONEXPANSIVE MAPPINGS[1]

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1. Introduction.

1.1. A mapping $f: X \to X$ of a metric space X into itself is said to be nonexpansive (ϵ-nonexpansive) if the condition

$$(1) \qquad d(f(p), f(q)) \leqq d(p, q)$$

holds for all $p, q \in X$ (for all p, q with $d(p, q) < \epsilon$).

Mappings, as above, satisfying (1) with the strict inequality sign for all $p, q \in X$, $p \neq q$ (for all p, q with $0 < d(p, q) < \epsilon$) are called contractive (ϵ-contractive).

A point $y \in Y \subset X$ is said to belong to the f-closure of Y, $y \in Y^f$, if $f(Y) \subset Y$ and there is a point $\eta \in Y$ and a sequence $\{n_i\}$ of positive integers, $(n_1 < n_2 < \cdots < n_i < \cdots)$, so that $\lim f^{n_i}(\eta) = y$.

1.2. In [1] we proved that a point of X^f is fixed if f is contractive, periodic if f is ϵ-contractive. The corresponding statements for nonexpansive and ϵ-nonexpansive mappings do not hold unless some additional assumption is made either on X or on f. We prove (Theorems 1, 1′) that a weaker conclusion involving suitable generalizations of fixed and periodic points is still true.

To this end we introduce the notions of isometric and ϵ-isometric sequences.

1.3. A sequence $\{x_i\} \subset X$ is said to be an isometric (ϵ-isometric) sequence if the condition

$$(2) \qquad d(x_m, x_n) = d(x_{m+k}, x_{n+k})$$

holds for all $k, m, n = 1, 2, \cdots$; (for all $k, m, n = 1, 2, \cdots$; with $d(x_m, x_n) < \epsilon$). A point $x \in X$ is said to generate an isometric (ϵ-isometric) sequence, under f, if $\{f^n(x)\}$ is such a sequence.

1.4. In the special case when $X = E^n$, ϵ-nonexpansive mappings are actually nonexpansive. In this case $X^f \neq \phi$ implies the existence of fixed points (Theorem 2).

2. Nonexpansive mappings in general metric spaces.

2.1. Proposition 1. *If $f: X \to X$ is ϵ-nonexpansive and $x \in X^f$ then a sequence $\{m_j\}$, $(m_1 < m_2 < \cdots < m_j < \cdots)$, of positive integers exists so that* $\lim_{j\to\infty} f^{m_j}(x) = x$. *[Hence, in particular, $(X^f)^f = X^f$.]*

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PROOF. By the definition of x a point $\xi\in X$ and a sequence $\{n_i\}$ exist so that $\lim_{i\to\infty} f^{n_i}(\xi)=x$. If $f^m(\xi)=x$ for some m then $\{m_j\} =\{n_i-m\}$ $(n_i>m)$, is a sequence as desired. Otherwise let δ, $0<\delta <\epsilon$, be an arbitrary, but fixed, number. Then an $i=i(\delta)$ exists so that

$$(3) \qquad d(x,f^{n_{i+j}}(\xi)) < \frac{\delta}{4} \qquad (j = 0, 1, 2, \cdots).$$

For such an i and for arbitrary k $(k=1, 2, \cdots)$,

$$(4) \qquad d(f^{n_i}(\xi),f^{n_{i+k}}(\xi)) \leqq d(x,f^{n_i}(\xi)) + d(x,f^{n_{i+k}}(\xi)) < \frac{\delta}{2} \cdot$$

After $n_{i+k}-n_i$ iterations performed on both x and $f^{n_i}(\xi)$ we obtain from (3) (taken in the case $j=0$) and (1)

$$d(f^{n_{i+k}-n_i}(x),f^{n_{i+k}}(\xi)) < \frac{\delta}{4} \cdot$$

Hence

$$d(x,f^{n_{i+1}-n_i}(x)) \leqq d(x,f^{n_i}(\xi)) + d(f^{n_i}(\xi),f^{n_{i+1}}(\xi)) + d(f^{n_{i+1}}(\xi),f^{n_{i+1}-n_i}(x))$$
$$< \frac{\delta}{4}+\frac{\delta}{2}+\frac{\delta}{4} = \delta.$$

We put $m_1=n_{i+1}-n_i$.

Suppose, now, that $m_1<m_2< \cdots <m_{j-1}$ are already defined and that $d(x, f^{m_i}(\xi))\leqq\frac{1}{2}\min_{m=1,\cdots,m_{i-1}} d(x, f^m(\xi))$ $(i=2, 3, \cdots, j-1)$. Then we choose $m_j=n_{l+1}-n_l$, where l is chosen so as to satisfy (3) with δ replaced by $\frac{1}{2}\min_{m=1,\cdots,m_{j-1}} d(x,f^m(\xi))$. It is readily seen that the sequence $\{m_j\}$ so defined satisfies the requirements of the proposition.

2.2. THEOREM 1. *If $f\colon X\to X$ is an ϵ-nonexpansive mapping of X into itself then each $x\in X^f$ generates an ϵ-isometric sequence.*

PROOF. Suppose that contrary to the assertion of the theorem indices m, n and k exist so that

$$(5) \qquad d(f^m(x),f^n(x)) < \epsilon$$

and $\delta=d(f^m(x),f^n(x))-d(f^{m+k}(x),f^{n+k}(x))\neq 0$. By (1), then,

$$(6) \quad d(f^m(x),f^n(x)) - d(f^{m+l}(x),f^{n+l}(x)) \geqq \delta > 0 \ (l = k, k + 1, \cdots).$$

Also from (1) and Proposition 1 it follows that for some $\{n_j\}$ and all l

$$\lim_{j\to\infty} f^{n_j}(f^l(x)) = \lim_{j\to\infty} f^{n_j+l}(x) = f^l(x). \tag{7}$$

Hence a positive integer i exists such that $j \geqq i$ implies

$$d(f^{m+n_j}(x), f^m(x)) < \frac{\delta}{2},$$

$$d(f^{n+n_j}(x), f^n(x)) < \frac{\delta}{2}.$$

However,

$$\begin{aligned} d(f^m(x), f^n(x)) &\leqq d(f^m(x), f^{m+n_j}(x)) + d(f^{m+n_j}(x), f^{n+n_j}(x)) \\ &\quad + d(f^{n+n_j}(x), f^n(x)) < \frac{\delta}{2} + d(f^{m+n_j}(x), f^{n+n_j}(x)) + \frac{\delta}{2} \end{aligned}$$

which is incompatible with (6) for $n_j \geqq \max(n_i, k)$. This contradiction shows that $\delta = 0$, thus establishing Theorem 1.

The corresponding statement concerning nonexpansive mappings (an immediate consequence of Theorem 1) is stated next.

THEOREM 1$'$. *If $f: X \to X$ is nonexpansive and $x \in X^f$ then x generates an isometric sequence.*

2.3. **Remarks.**

2.31. A somewhat stronger formulation of Theorems 1 and 1$'$ is feasible. It follows, namely, from Proposition 1 that it suffices to assume that $f | X^f$ is ϵ-nonexpansive (nonexpansive).

2.32. If f is contractive (ϵ-contractive) and x generates an isometric sequence ($x \in X^f$ generates an ϵ-isometric sequence) then it is readily seen that x is a fixed (periodic) point. This shows that Theorems 1 and 1$'$ are generalizations of their counterparts in [1].

2.33. If $f: X \to X$ is a nonexpansive mapping having a fixed point at, say, $\omega \in X$ then each isometric sequence generated by a point of X^f lies on a sphere centered at ω. This assertion can be seen to follow from an argument analogous to that used in the proof of Theorem 1.

2.34. In some spaces an ϵ-nonexpansive mapping is also nonexpansive. This is always the case when X is ϵ-chainable and the condition

$$d(p, q) = \inf_{C(p,q)} \sum_{i=1}^{n} d(x_{i-1}, x_i),$$

where $C(p, q)$ denotes the collection of all ϵ-chains $p = x_0, x_1, \cdots, x_n = q$, ($n$ arbitrary, $d(x_{i-1}, x_i) < \epsilon$), holds. Indeed, since f is ϵ-nonexpansive we have

$$d(f(x_{i-1}),\ f(x_i)) \leqq d(x_{i-1}, x_i) \quad \text{provided} \quad d(x_{i-1}, x_i) < \epsilon.$$

Hence

$$d(f(p), f(q)) \leqq \inf_{C(p,q)} \sum_{i=1}^{n} d(f(x_{i-1}), f(x_i))$$

$$\leqq \inf_{C(p,q)} \sum_{i=1}^{n} d(x_{i-1}, x_i) = d(p, q)$$

for all p, q, as asserted.

3. Nonexpansive mappings in E^n.

3.1. THEOREM 2. *Let $f: E^n \to E^n$ be nonexpansive and $(E^n)^f \neq \phi$. Then:*

(a) *there is a $\xi \in E^n$ such that $f(\xi) = \xi$;*

(b) *if $x \in (E^n)^f$ and V is the linear variety of smallest dimension containing $\{f^m(x)\}$ then V contains a unique fixed point.*

Several properties of nonexpansive mappings, needed for the proof of the above theorem, will be given in the propositions and lemmas that follow.

PROPOSITION 2. *Let $f: E^n \to E^n$ be nonexpansive. If the restriction, $f|A$, of f to a subset A of E^n is an isometry then the restriction, $f|\mathrm{co}\ A$, of f to the convex hull, $\mathrm{co}\ A$, of A, is an isometry too; i.e.,*

$$d(c_1, c_2) = d(f(c_1), f(c_2)) \qquad (c_1, c_2 \in \mathrm{co}\ A). \tag{8}$$

Furthermore, if, in addition, $f(A) \subset A$ then

$$f(\mathrm{co}\ A) \subset \mathrm{co}\ A. \tag{9}$$

PROOF. To prove (8) it suffices to show that

$$d(c, a) = d(f(c), f(a)) \qquad (c \in \mathrm{co}\ A,\ a \in A). \tag{10}$$

Indeed, if (10) holds then $A_1 = A \cup \{c_1\}$ is mapped isometrically on $f(A_1)$ and (10) applies to $c_2 \in \mathrm{co}\ A_1 = \mathrm{co}(A \cup \{c_1\}) = \mathrm{co}\ A$ and $c_1 \in A_1$, yielding (8).

To prove (10) we use induction on the positive integer k in the representation $c = \sum_{i=1}^{k} \lambda_i a_i$, $0 < \lambda_i$, $\sum_{i=1}^{k} \lambda_i = 1$, $a_i \in A$. For $k = 1$, (10) holds by assumption. Suppose it is true for $k-1$ and write

$$c = \sum_{i=1}^{k-1} \lambda_i a_i + \lambda_k a_k = \alpha b_1 + (1 - \alpha) b_2,$$

where

$$\alpha = \sum_{i=1}^{k-1} \lambda_i, \qquad b_1 = \alpha^{-1} \sum_{i=1}^{k-1} \lambda_i a_i, \qquad b_2 = a_k.$$

From the fact that $b_2 \in A$ and b_1 satisfies (10) it follows that the triangles $\triangle a b_1 b_2$ and $\triangle f(a)\, f(b_1)\, f(b_2)$ are congruent; in particular $d(f(b_1), f(b_2)) = d(b_1, b_2)$. As c belongs to the segment (b_1, b_2) we have

$$\begin{aligned} d(c, b_1) &= d(b_1, b_2) - d(c, b_2) \leqq d(f(b_1), f(b_2)) - d(f(c), f(b_2)) \\ &\leqq d(f(b_1), f(c)) \leqq d(b_1, c). \end{aligned}$$

It follows that $\triangle a\, b_1\, c$ and $\triangle f(a)\, f(b_1)\, f(c)$ are congruent, whence (10). To prove (9) let S be an arbitrary simplex, with vertices in A containing c. Clearly $f(S)$ is a simplex contained in co A; hence $f(c) \in f(S) \subset \text{co } A$.

3.2. Remark. It can be readily seen that if in (9) and (10) co A is replaced by the closed convex hull cl co A then the so-obtained relations $(9)^-$ and $(10)^-$ remain valid.

3.3. Proposition 3. *Suppose $f: E^n \to E^n$ is nonexpansive and maps a closed convex subset $C \subset E^n$ into itself. If $f(x) = x$ for some $x \in E^n$ then a point $y \in C$ exists with $f(y) = y$.*

Proof. By a well-known property of closed convex sets a point $y \in C$ exists with $d(x, y) = \min_{c \in C} d(x, c)$. Let $B = \{ b \mid d(b, x) \leqq d(x, y) \}$. Clearly $B \cap C = \{ y \}$ and, by (1), $f(B) \subset B$. Hence $f(y) \in f(B) \cap f(C) \subset B \cap C = \{ y \}$; i.e., $f(y) = y$ as asserted.

3.4. Lemma 1. *Let $g: A \to A$ be an isometry of $A \subset E^n$ and let V be the linear variety of smallest dimension containing A. Then a mapping $g^*: V \to V$ exists which is an isometry on the whole of V, and $g^*(a) = g(a)$ for all $a \in A$.*

The straightforward verification of Lemma 1 is omitted.

3.5. Lemma 2. *Let $h: E^n \to E^n$ be an isometry with $(E^n)^h \neq \phi$. Then there is a $\xi \in E^n$ such that $h(\xi) = \xi$.*

Proof. Since an isometry of E^n is a rigid motion it is the composition of an orthogonal transformation and a translation.

We introduce into E^n an orthonormal system of coordinates with respect to which the matrix M of the orthogonal transformation is in normal form. If $x = (x_1, x_2, \cdots, x_n)$ and $h(x) = y = (y_1, y_2, \cdots, y_n)$, then

$$y = xM + a,$$

where

$$M = \begin{pmatrix} I_{n_1} & & \\ & -I_{n_2} & \\ & & R \end{pmatrix}, \tag{11}$$

with

$$I_{n_1} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} n_i, \qquad R = \begin{pmatrix} R_1 & & & \\ & R_2 & & \\ & & \ddots & \\ & & & R_{1/2(n-n_1-n_2)} \end{pmatrix},$$

$$R_j = \begin{pmatrix} \cos\phi_j & -\sin\phi_j \\ \sin\phi_j & \cos\phi_j \end{pmatrix},$$

$$\phi_j \neq \nu\pi \quad \left(j = 1, 2, \cdots, \frac{n - n_1 - n_2}{2}; \ \nu = 0, 1, 2, \cdots\right)$$

and $a = (a_1, a_2, \cdots, a_n)$.

From the assumption that $(E^n)^h \neq \phi$ it now follows that $a_i = 0$ for all $i = 1, 2, \cdots, n_1$. Indeed should any of the a_i's, say a_1, be different from zero then the sequence of hyperplanes satisfying $x_1 = na_1$, $n = 1, 2, \cdots$, would separate between successive iterates of x, under h; hence no $\{h^m(x)\}$ could have a convergent subsequence. A direct computation readily shows that $\xi = (\xi_1, \cdots, \xi_{n_1}; \xi_{n_1+1}, \cdots, \xi_{n_1+n_2}; \xi_{n_1+n_2}, \cdots, \xi_n)$, where

$$\xi_i = \begin{cases} \text{arbitrary} & \text{for } i = 1, 2, \cdots, n_1 \\ \dfrac{a_i}{2} & \text{for } i = n_1 + 1, \cdots, n_1 + n_2, \\ \left(4\sin^2\dfrac{\phi_k}{2}\right)^{-1} \begin{vmatrix} \sin\phi_k & a_i \\ \cos\phi_k - 1 & a_{i+1} \end{vmatrix} & \text{for } i = n_1 + n_2 + 2k - 1, \\ & \left(k = 1, 2, \cdots, \dfrac{n - n_1 - n_2}{2}\right) \\ \left(4\sin^2\dfrac{\phi_k}{2}\right)^{-1} \begin{vmatrix} a_i & \cos\phi_k - 1 \\ a_{i+1} & -\sin\phi_k \end{vmatrix} & \text{for } i = n_1 + n_2 + 2k, \end{cases}$$

remains fixed under h. The fact that no ϕ_j is a (even) multiple of π implies that there is at least one such ξ thus proving the lemma.

3.6. Proof of Theorem 2. Let $x \in (E^n)^f$. Then the restriction

$f|\{f^m(x)\}$ is obviously an isometry. By $(9)^-$ and $(10)^-$, $g=f|\operatorname{cl}\operatorname{co}\{f^m(x)\}$ maps isometrically $\operatorname{cl}\operatorname{co}\{f^m(x)\}$ into itself. By Lemma 1, there is an extension g^* of g which maps isometrically the linear variety V of smallest dimension that contains $\{f^m(x)\}$ onto itself. Obviously g^* can be extended to an isometry $h: E^n \to E^n$. Lemma 2 and Proposition 3 apply to the effect that a $\xi \in \operatorname{cl}\operatorname{co}\{f^m(x)\}$ exists such that $h(\xi)=\xi$. Hence $f(\xi)=g(\xi)=g^*(\xi)=h(\xi)=\xi$.

That ξ is the only fixed point in V, under f, can be seen by an application of 2.33. Suppose that $\eta \in V$ is another fixed point. Then $\{f^m(x)\}$ is contained in the intersection of two spheres lying in V and centered at ξ and η respectively. This implies, however, that $\{f^m(x)\}$ is contained in a linear subvariety of V, against the definition of V. This proves (b) and, thus, accomplishes the proof of Theorem 2.

REMARK. The proof of Theorem 2 and the discussion preceding it actually show that the following stronger statement holds. If $f: A \to A$, $A \subset E^n$, is nonexpansive, $A^f \neq \phi$ and there is an $x \in A^f$ such that $\operatorname{cl}\operatorname{co}\{f^m(x)\} \subset A$ then a unique $\xi \in \operatorname{cl}\operatorname{co}\{f^m(x)\}$ satisfies $f(\xi)=\xi$.

REFERENCE

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