ON CLANS OF NON-NEGATIVE MATRICES

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A *clan* is a compact connected topological semigroup with identity. Professor A. D. Wallace has raised the following question [**9**]: Is a clan of real $n\times n$ matrices with non-negative entries, which contains the identity matrix, necessarily acyclic? That is to say, do all of the Alexander-Čech cohomology groups with arbitrary coefficients (in positive dimensions) vanish? In this paper the slightly stronger result, that any non-negative matrix clan is contractible, is obtained. This follows from the result, interesting in itself, that a compact group of non-negative matrices is finite (Theorem 2).

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The set of order n non-negative matrices is denoted by N_n. The real and complex general linear groups of order n are represented by $\mathrm{Gl}(n, R)$ and $\mathrm{Gl}(n, C)$, respectively. The semigroup terminology used is that of [**8**]; in particular, K denotes the minimal ideal of a clan S, E denotes the set of idempotents of S, and for $e\in E$, $H(e)$ is the maximal subgroup of S containing e. An iseomorphism is an isomorphism which is also a homeomorphism. The topology of N_n is any locally convex topology; for example, the topology of Euclidean n^2-space.

The equation $M=\mathrm{diag}(A, B)$ means that M is the matrix which, in 2×2 block form, has the square submatrix A in the upper left corner, the square submatrix B in the lower right corner, and zero entries elsewhere. The $k\times k$ identity matrix is denoted by I_k when used as a submatrix. The set of eigenvalues of a matrix M is denoted by $S(M)$.

The well-known theorem [**1**, p. 80] that a non-negative matrix M has a real eigenvalue r such that if $\lambda\in S(M)$, then $|\lambda|\leqq r$ is used without proof. Also used without proof is the following theorem, due to Karpelevich [**3**], and stated in less than full generality:

THEOREM 1. *Let $M\in N_n$, and let M have maximal real eigenvalue* 1. *If $\lambda\in S(M)$, $|\lambda|=1$, then $\lambda^k=1$ for some $k\leqq n$.*

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LEMMA 1. *Let $X \in G$, a compact subgroup of* Gl(n, C). *If* $\lambda \in S(X)$, *then* $|\lambda| = 1$.

PROOF. The determinant function maps G homomorphically into the unit circle. Hence $1 = |\det X| = |\lambda_1 \lambda_2 \cdots \lambda_n|$, $\lambda_i \in S(X)$. Let $P \in \mathrm{Gl}(n, C)$ such that $A = PXP^{-1}$ is triangular, diagonal $A = (\lambda_1, \lambda_2, \cdots, \lambda_n)$. Since diagonal $A^t = (\lambda_1^t, \lambda_2^t, \cdots, \lambda_n^t)$ and the group PGP^{-1} is compact, if follows that $|\lambda_i| \leqq 1, i = 1, \cdots, n$. This is clearly sufficient.

THEOREM 2. *Let $H(e)$ be a compact topological group, $H(e) \subset N_n$. Then $H(e)$ is finite.*

PROOF. Define $f: H(e) \rightarrow \mathrm{Gl}(n, R)$ by $f(x) = x + I - e$. The function f is clearly an iseomorphism. Since $f(H(e))$ is a compact subgroup of $\mathrm{Gl}(n, R)$, $H(e)$ is a Lie group. The identity component C of $H(e)$ is therefore open; hence it suffices to prove that $H(e)$ is totally disconnected. If $C \neq \{e\}$, then C has a nontrivial one parameter group [5, p. 105], hence elements of infinite order. The proof is then completed by contradiction when it is shown that every element of $H(e)$ has finite order.

Let $X \in H(e)$. There exists $B \in \mathrm{Gl}(n, R)$ such that $BeB^{-1} = \mathrm{diag}(I_k, 0)$, where rank e is assumed equal to k. Since BeB^{-1} is an identity for BXB^{-1}, $BXB^{-1} = \mathrm{diag}(X_k, 0)$, where X_k is a rank k real $k \times k$ matrix. Let f be the iseomorphism of $BH(e)B^{-1}$ into $\mathrm{Gl}(n, R)$ defined by $f(BXB^{-1}) = BXB^{-1} + I - BeB^{-1}$. Since $f(BH(e)B^{-1})$ is iseomorphic to $H(e)$, it suffices to find an integer m such that $f(BXB^{-1})^m = f(BeB^{-1}) = I$.

Assume $k < n$. Note $S(X) = S(BXB^{-1}) = S(f(BXB^{-1})) \cup \{0\}$. For if $\lambda \in S(BXB^{-1})$, $\lambda \neq 0$, then $\det(X_k - \lambda I_k) = 0$. Hence

$$\det(f(BXB^{-1}) - \lambda I) = (1 - \lambda)^{n-k} \cdot \det(X_k - \lambda I_k) = 0$$

and

$$\lambda \in S(f(BXB^{-1})).$$

Conversely, if $\lambda \neq 1$ and $\lambda \in S(f(BXB^{-1}))$, then $\lambda \in S(BXB^{-1})$. Finally, by Lemma 1, $\lambda \in S(f(BXB^{-1}))$ gives $|\lambda| = 1$; therefore $\lambda \in S(BXB^{-1})$, $\lambda \neq 0$ also yields $|\lambda| = 1$. Since $X \in N_n$, $1 \in S(BXB^{-1})$, and $S(BXB^{-1}) = S(f(BXB^{-1})) \cup \{0\}$. By Theorem 1, $S(BXB^{-1}) \subset \{\lambda : \lambda^t = 1, t \leqq n\} \cup \{0\}$. If $k = n$, a similar argument can be given. In either event $S(f(BXB^{-1})) \subset \{\lambda : \lambda^t = 1, t \leqq n\}$. Let $P \in \mathrm{Gl}(n, C)$ such that $D = Pf(BXB^{-1})P^{-1}$ is lower triangular and diagonal $D = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. Note $\lambda_i \in S(f(BXB^{-1}))$, $i = 1, \cdots, n$. Let $m =$ least common multiple $\{t_i : \lambda_i^{t_i} = 1, t_i \leqq n\}$. Then diagonal $D^m = \{1, 1, \cdots, 1\}$. Now if $j=i-1$, then $(D^{mp})_{ij}=p\cdot(D^m)_{ij}$. Hence, by the compactness of $Pf(BH(e)B^{-1})P^{-1}$, $(D^m)_{ij}=0$, $j=i-1$. By a straightforward induction, it follows that $(D^m)_{ij}=0, j<i, i=1, \cdots, n$. Hence $D^m=I$, and therefore $f(BXB^{-1})$ has order $\leqq m$, which completes the proof.

COROLLARY 1. *Let S be a continuum semigroup in N_n. Then $K\subset E$.*

PROOF. Fix $e\in E\cap K$. Then $eSe=H(e)$ [8]. Since eSe is a continuum, it is degenerate; hence $H(e)=\{e\}$. The corollary now follows from the fact that $K=\bigcup\{H(e): e\in K\}$.

If S is a clan, it is known [8] that $H^n(S)=H^n(eSe)$ for $e\in K\cap E$, $n\geqq 0$. If, also, $S\subset N_n$, then by Theorem 2, $H^n(S)=H^n(\{e\})=0$, $n>0$. Hence S is acyclic. It will now be shown that S is contractible. The following lemma is due to Gluskin [2].

LEMMA 2. *Let S be an $n\times n$ complex matrix semigroup. Let $e,f\in E$ and $f\in eSe$. If $f\neq e$, then* rank $f<$rank e.

PROOF. Suppose rank $e=r$, $e\neq f$. Choose v such that $vev^{-1}=\operatorname{diag}(I_r, 0)$. Then $vfv^{-1}=\operatorname{diag}(g, 0)$, since e is an identity for f. Note g is an $r\times r$ complex matrix, and $g^2=g$. Since rank $vfv^{-1}=$rank f, it suffices to show $\det(g)=0$. If this is not the case, then g is an idempotent in $\mathrm{Gl}(r, C)$; hence $g=I_r$. But this implies $f=e$, contrary to assumption. This completes the proof.

An *I-semigroup* is a clan on an interval such that one endpoint is an identity and the other a zero. It is shown in [6] that the only types of I-semigroups are the following: (i) S has the multiplication of the real interval $[0, 1]$; (ii) S has a multiplication isomorphic to the interval $[1/2, 1]$ under the operation $x\circ y=\max\{1/2, xy\}$; (iii) S is idempotent and has a multiplication isomorphic to the interval $[0, 1]$ under the operation $x\circ y=\min\{x, y\}$; (iv) S is the union of a collection of semigroups of types (i), (ii), and (iii) which meet only at their respective endpoints.

LEMMA 3. *Let S be a clan in which, for each $e\in E$, $H(e)$ is totally disconnected. Suppose also that there exists a neighborhood V of* 1 *such that $V\cap E=1$. Then there is an I-semigroup in S having* 1 *as an identity.*

PROOF. It is well known [7] that the existence of the neighborhood V above is sufficient to insure a local one-parameter semigroup $\sigma([0, 1])$ in V such that $\sigma(0)=1$, $\sigma(a)\notin H(1)$, $0<a\leqq 1$, and if $\sigma(a)=\sigma(b)g$, $g\in N(1)$, then $a=b$ and $g=1$. In the same paper, it is shown that σ can be extended to a full one-parameter semigroup by defining $\sigma(t)=\sigma(1)\sigma(t-1)$ for $t\in[1, 2]$ and proceeding inductively. Now the closure of $\sigma([0, \infty))$ is a commutative clan, hence its minimal ideal is

a connected group, and therefore a single point. It follows by a theorem of Koch [4] that this clan has exactly 2 idempotents and is an I-semigroup.

THEOREM 3. *Let S be a nondegenerate clan in N_n. Then S contains an I-semigroup from* 1 *to K, and S is contractible.*

PROOF. By Lemma 2, there exists a neighborhood V of 1 containing no other idempotents; this follows from the fact that the rank of an idempotent equals its trace. By Theorem 2 each $H(e)$ is finite. It follows from Lemma 3 that there exists an I-semigroup from 1 to $e \in E$. By Lemma 2, rank $e <$ rank 1. If $e \notin K$, then eSe is a nondegenerate subclan with identity e, and the above argument produces an I-semigroup from e to $f \in E$, rank $f <$ rank e. In this manner, an idempotent of minimal rank in S is obtained, which clearly belongs to K. The union of the I-semigroups constructed above is the desired I-semigroup.

Let T be an I-semigroup in S with endpoints 1 and $e \in K \cap E$. Define $F: S \times T \to S$ by $F(x, t) = txt$. Then $F(x, 1) = x$, and $F(x, e) = exe = e$, for each $x \in S$. Hence S is contractible. This completes the proof.

By Lemma 2, no I-semigroup in N_n can be of type (iii) mentioned above. On the other hand, it is well known that if A is a nilpotent $n \times n$ complex matrix, then $A^n = 0$. It follows that the I-semigroups in N_n are either of type (i), or of type (iv), constructed by joining together the endpoints of semigroups of type (i).

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